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# Do we really need a 'contingency model' for concept formation? A reply to Richardson & Bhavnani (1984)

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In a recent paper, Richardson & Bhavnani (1984) proposed a new model of concept formation christened 'the contingency model' which would supersede the so-called 'prototype model'. To support their model, they ran an experiment in which the contingency model apparently surpassed a prototype model. It is argued here that their conclusions are doubtful for several reasons. Firstly, some miscomputations have slipped into the data analysis. Secondly, the prototype model used is not appropriate. Precisely, Richardson & Bhavnani have computed their 'Contingency Values' (CV) assuming that the features composing the exemplars are nominal variables, but have computed one 'distance to prototype' assuming the features are ordinal variables. The alleged superiority of the contingency model vanishes when a correct analysis is performed, or when it is evaluated against what we call a 'mimimal prototype model' in which the features are nominal variables. The 'minimal prototype model' can be interpreted as a particular and simple case of most of the current 'prototype models' (recalling that there is not one but several models). Consequently, it provides an overall test between the CV model and most of the current prototype models. The minimal model is shown to be a particular case of the 'distance to prototype' models, and the 'family resemblance to exemplars models' (including Tversky's contrast model), and the 'distributed memory models'. Finally, we note that the contingency model (as tested by Richardson & Bhavnani) is a variant of a 'distance to prototype model', in which the distance from the prototype to an exemplar is given by an entropic measure. Thus the opposition proposed by Richardson & Bhavnani between the CV model and the prototype models is meaningless.

Since the early work of Moore (1910) and Fisher (1916), psychology has produced numerous theories of concept formation. It is customary to distinguish the so-called classical approach (cf. Hull, 1920; Bruner *et al.*, 1956; Bourne, 1966) from the 'prototype models' (sometimes called the 'probabilistic' approach). This latter has led recently to an abundance of empirical research and models (for some reviews see Reed, 1978; Rosch, 1978, 1983; Millward, 1980; Cordier & Dubois, 1981; Mervis & Rosch, 1981; Smith & Medin, 1981; Lakoff, 1982; Estes, 1983; Eckes & Six, 1984; Homa, 1984; Medin & Smith, 1984; Abdi, 1986; for a bibliography see Abdi, 1985). In a recent paper, Richardson & Bhavnani (1984) proposed a model which not only adds a new species to this luxuriance, but claims to render all the other models obsolete. In their paper, the authors first recalled some problems faced by the 'prototype models' (in fact, essentially the fuzzy-set formulation; cf. Osherson & Smith, 1981, 1982; Armstrong *et al.*, 1983; Cohen & Murphy, 1984; Smith & Osherson, 1984); and, second, proposed a new model: the 'contingency model'. They argued that their model is superior to the 'prototype model' (implying, incidentally, that there is only one such model), and supported their claim by an experiment that supposedly contrasts the two models. They reported a significant advantage for the contingency model. However, when correctly analysed, this experiment *does not support* their position.

### The contingency model of Richardson & Bhavnani (1984)

In this section, we review the model of Richardson & Bhavnani (1984) as they submitted it to test (cf. the Experiment section). A more formal presentation is given in Appendix A. They tested the general idea that in acquiring a concept we abstract some information from the family of exemplars presented during the learning part of the experiment. When presented with a new exemplar to evaluate, we decide to assign it (or not) to a category by using the probability (computed under the 'independence hypothesis') of observing this exemplar.

Specifically, assume that an exemplar of a concept is a set of nominal variables, each variable having a fixed and finite number of exclusive levels (i.e. an exemplar can have only *one* level of each variable).

The probability of observing a particular level of one nominal variable is obtained by dividing the number of occurrences of this level by the total number of exemplars (e.g. the probability of observing the level 2 for the first variable is $9/29 = 0{\cdot}310$, in Table 1). *Under the independence hypothesis*, the probability of observing any composed event is computed as the product of the probabilities of the elementary events. Thus the probability of observing the object 22433 is computed – under the independence hypothesis – as the product of the probability of observing each of its features, that is:

prob(22433) = prob(2 in first position) * prob(2 in second position) * prob(4 in third position) * prob(3 in fourth position) * prob(3 in fifth position)

$$= 9/29 * 8/29 * 6/29 * 8/29 * 9/29 = 0{\cdot}152 * 10^{-2}.$$

Richardson & Bhavnani (1984) called this probability the 'contingency value', and proposed (in their experiment section) that the greater the contingency value of an exemplar, the greater is the probability that this exemplar will be falsely recognized. Richardson & Bhavnani then argued that the so-called prototypicality effect is simply explicable by a confounding between typicality measure (e.g. distance to prototype) and contingency value.

To support their claim, Richardson & Bhavnani (1984) designed an experiment in which they contrasted the contingency model with a prototype model. In this experiment, 25 subjects were first presented a set of 29 'to-be-learned' schematic faces, each composed of five nominal variables (face length, eye size, nose length, mouth width, number of bilateral cheek lines). Each variable has four levels, and a face can be described by its formula, e.g. 11242, 12314. After completion of the acquisition phase, the subjects were then presented with a set of 10 new schematic faces and asked to give a signed score of recognition confidence: from $-5$ if they were very sure of not having seen the face, to $+5$ if they were very sure of having seen it. From these data a mean confidence rating is computed. Richardson & Bhavnani compared – using multiple regression on the ranked values – the quality of the prediction of the contingency model with the prediction of a prototype model. They reported a significant advantage of the CV model over the prototype model.

It is shown, here, that their conclusion is invalid for at least two methodological reasons:

1. A minor one: the computation of the contingency values and the statistical analysis are incorrect.
2. A major one: the distance from the prototype of the exemplars to the test set of faces is miscomputed (i.e. there is a confounding of variables).

When these errors are corrected the alleged advantage of the contingency model vanishes.

A third major criticism, namely that the contingency model, as tested by Richardson & Bhavnani, is simply a particular case of an 'entropic' distance to the prototype model, is left for the Discussion section.

## Methodological rectifications

### *Statistical analysis*

Two problems appear in the statistical analysis performed by Richardson & Bhavnani. First, the contingency values given in Table 3 of their paper are miscomputed.*

* A referee (commenting on this paper) has suggested a possible and astute explanation for the errors in the computations of the CV values. According to the referee, Richardson & Bhavnani have made two errors. Firstly they have divided each contingency by 30 instead of 29. Secondly, they have only considered the first 2 decimal

**Table 1.** Contingency table derived from Richardson & Bhavnani (1984) 29 exemplars learning set. The rows stand for the levels of the nominal variables composing the faces. The columns stand for the nominal variables. At the intersection of a row and a column, the number of exemplars having the row-level for the column-variable is found, e.g. 7 faces have the level 1 for the first variable.

| | Nominal variables composing the learning set | | | | |
|---|---|---|---|---|---|
| Levels of the variables | 1 | 2 | 3 | 4 | 5 |
| 1 | 7 | 9 | 9 | 5 | 7 |
| 2 | 9 | 8 | 7 | 10 | 6 |
| 3 | 5 | 7 | 7 | 8 | 9 |
| 4 | 8 | 5 | 6 | 6 | 7 |
| | 29 | 29 | 29 | 29 | 29 |

Secondly, some errors have slipped into the multiple regression analysis. The correct results are given in Table 2. To evaluate the different models, Richardson & Bhavnani used multiple regression on the ranked values (i.e. they performed a Spearman correlation): the correct correlation matrix is given in Table 3. Both for ranked values and raw scores the correlation between the mean recognition confidence rating and the CV values is greater than the correlation between the mean recognition and 'the distance to prototype' computed by Richardson & Bhavnani:

raw scores: $MS_{additional} = 4{\cdot}2423$; $r^2_{additional} = 0{\cdot}132$; $F = 2{\cdot}14$, d.f. $= 1, 7$, n.s.
ranked scores: $MS_{additional} = 25{\cdot}88$; $r^2_{additional} = 0{\cdot}314$; $F = 6{\cdot}65$, d.f. $= 1, 7$, $P < 0{\cdot}05$.

However, the advantage reaches significance *only for transformed data*, although no theoretical reason can be given to justify any transformation of the data for the statistical analysis (cf. Abdi, in press).

### *Correct prototype models*

As can be seen in Table 2, the distance to prototype is computed assuming that the variables describing the pseudo-faces are ordinal (i.e. the distance between, say, 11111 and 11113 is 2, not 1), but the same variables are seen as nominal when used by the contingency model. So, the comparison between the two models is not appropriately conducted, especially since there is some strong evidence indicating that subjects evaluate the kind of variables used by Richardson & Bhavnani as nominal ones rather than ordinal ones (Neumann, 1974, 1977; Homa & Vosburgh, 1976; Solso & McCarthy, 1981). Obviously, in order to compare the models, the same decision concerning the variables describing the exemplars has to be taken (i.e. nominal or ordinal in both cases). By not doing so Richardson & Bhavnani have tested a nominal model against an ordinal model rather than a CV model against a prototype model. Consequently, the alleged advantage of the CV model simply reflects this confounding. Although an ordinal version of the CV

places (i.e. all rounding goes down). So for the exemplar 22433 used as an example here Richardson & Bhavnani calculated the CV value to be:

$$CV = 9/30 * 8/30 * 6/30 * 8/30 * 9/30 = 0{\cdot}128 * 10^{-2}$$
$$= 0{\cdot}120 * 10^{-2}$$

Bizarre as this seems it is consistent with all the target exemplars in Table 3 of Richardson & Bhavnani.

**Table 2.** Corrected Table 3 (for CV values) of Richardson & Bhavnani (1984)

| Exemplar | Rating | Rank | CV*100 | Rank | Distance from prototype | Rank |
|---|---|---|---|---|---|---|
| 22433 | 4·20 | 1 | 0·152 | 2 | 5 | 5 |
| 44242 | 0·12 | 7 | 0·049 | 8·5 | 9 | 8·5 |
| 22211 | 2·64 | 3 | 0·086 | 4 | 5 | 5 |
| 42411 | 1·72 | 5 | 0·066 | 7 | 9 | 8·5 |
| 11223 | 2·08 | 4 | 0·194 | 1 | 2 | 1 |
| 11211 | 0·08 | 8 | 0·075 | 5 | 5 | 5 |
| 22212 | 1·40 | 6 | 0·074 | 6 | 4 | 3 |
| 22222 | 3·36 | 2 | 0·147 | 3 | 3 | 2 |
| 44444 | −1·37 | 10 | 0·049 | 8·5 | 11 | 10 |
| 33342 | −1·36 | 9 | 0·043 | 10 | 8 | 7 |

**Table 3.** Correlation matrices for the data of Table 2

| | CV | Dist. | Rating |
|---|---|---|---|
| (a) *Raw scores correlation* | | | |
| CV | — | 0·789 | 0·746 |
| Distance | | — | 0·662 |
| Rating | | | — |
| (b) *Ranked scores correlation* (*Spearman*) | | | |
| CV | — | 0·793 | 0·815 |
| Distance | | — | 0·597 |
| Rating | | | — |

model can easily be constructed by using the 'weak-ordinal information measures' of Abdi (1980) and Abdi *et al.* (1980), the aforementioned references suggest we should instead treat the variables as nominal in both cases.

This is the topic of this section. However, it should be noted first, that there exist not one but several 'prototype models' (cf. the reviews cited in the introduction). Consequently the problem of choosing a prototype model has to be faced. To avoid such a dilemma, a 'minimal prototype model' is proposed that could be considered as a particular case of most of the models in the literature. The main *raison d'être* of this minimal model is to allow an overall comparison between the 'contingency model' and most of the current 'prototype models' *when the variables describing the faces are taken as nominal variables.* So, if the contingency model does not surpass this minimal model it would not surpass most of the current prototype models (because the different models having more free parameters will fit the data more closely). In a sense our minimal model can simply be envisaged as the common part of most of the current prototype models (the main excluded family being the 'fuzzy models', but as pointed out in the introduction these models are not currently in favour).

Most of the current models can be subsumed under three main approaches: firstly 'distance to the prototype models'; secondly 'distributed memory models'; thirdly 'family resemblance to exemplars models'. Our minimal model had to be compatible with these main approaches.

To begin with the distance to prototype approach (a more formal presentation is given in Appendix B), recall that the subjects are presented, during the concept acquisition phase, with a family of exemplars of the to-be-learned concept. Each exemplar is composed of a set of nominal variables, each of these being in turn composed of a set of mutually exclusive features or levels. Denote by $M$ the number of features or levels for all the variables; i.e. $M$ is the sum for all the variables of the number of levels (e.g. for the exemplars of Richardson & Bhavnani, $M = 4$ levels $*$ 5 variables $= 20$). Then each possible exemplar could be represented by a sequence of $M$ ones or zeroes, where 1 means that the exemplar possesses the feature and 0 that is does not. More formally, this sequence could be represented by an $M$-components vector (see Appendix B). For this exposition, we call $g$ the average vector whose components are the relative frequencies of the features (i.e. each component is obtained as the number of exemplars possessing the feature divided by the total number of exemplars).

$g$, being an average vector, is a prototype. Precisely, $g$ is a centre of gravity of the vectors describing the exemplars. Consequently, a 'distance to the prototype model' would suggest that a new exemplar is evaluated according to its distance from $g$. That is to say, the 'typicality' of an exemplar $e$ is a function of the distance from $e$ to $g$ as computed by the usual Euclidean formula.

This model is based on prototype abstraction and thus is compatible with the different 'distance to prototype models' [cf. among others: Posner & Keele, 1968, 1970; Rankin *et al.*, 1970; Reed, 1972; Smith & Medin, 1981; Homa, 1984; Smith & Osherson, 1984; and Cohen & Murphy's (1984) model if one identifies their notion of role with the nominal variables and their role-value with the levels]. As detailed in Appendix D, this model can be interpreted as an instance of the 'distributed memory models' which are currently gaining both cognitive and physiological evidence in their favour (cf. Anderson, 1977; Wood, 1978; Murdock, 1979, 1982, 1983; Anderson & Mozer, 1981; Kohonen *et al.*, 1981; Eich, 1982, 1985; Knapp & Anderson, 1984; Kohonen, 1984; Pike, 1984; McClelland & Rumelhart, 1985).

This model, however, is also compatible with the set of the 'features models' and 'family resemblance to the learning set models' (cf. among others: Rosch, 1975; Tversky, 1977; Smith & Medin, 1981). In particular, the minimal prototype model is an instance of Tversky's contrast model (in which only the intersection of features is taken into account). To see how, it suffices to define the resemblance between two exemplars as the number of common features. The resemblance between an exemplar and a set of exemplars (e.g. a learning set) is obtained as the sum of the resemblance between this exemplar and all the exemplars of the set. Although the 'distance to prototype' approach and the 'resemblance to a set of exemplars' approach seem at first sight different, they are completely equivalent (cf. Appendix C). Precisely, the 'family resemblance' is simply a linear function of the squared distance to the centre of gravity. Incidentally, this restates that the distinction between an exemplar model, a distance model and a featural model (cf. Smith & Medin, 1981) may be often more a matter of taste than a real issue (as previously pointed out by Hollan, 1975; Tversky, 1977; Hintzman & Ludlam, 1980). So the 'minimal model' can be seen as a particular case of most of the current models of typicality.

In sum, the 'minimal prototype model' can be interpreted as a 'distance model' or as a 'featural model', or as an 'exemplar model', or as a 'distributed memory model' depending upon the preference of the reader. Also, since this model is a particular case of all these models, the evidence supporting them also supports it. Hence the minimal prototype model can be used as a very convenient (and prophylactic) way of evaluating any potential new theory of 'categories and concepts'. Clearly, if a candidate is unable to fare better than the minimal prototype model, it will be surpassed by the current models

**Table 4.** Results of the comparison between the contingency model and the 'minimal prototype model'

| Exemplar | $d^2$ $(e, g)$ in a 20-dimension Euclidean space | Resemblance to the family | CV * 100 | Rating |
|---|---|---|---|---|
| 22433 | 3·54 | 40 | 0·152 | 4·20 |
| 44242 | 4·09 | 32 | 0·049 | 0·12 |
| 22211 | 3·81 | 36 | 0·086 | 2·64 |
| 42411 | 3·95 | 34 | 0·066 | 1·72 |
| 11223 | 3·40 | 42 | 0·194 | 2·08 |
| 11211 | 3·89 | 35 | 0·075 | 0·08 |
| 22212 | 3·89 | 35 | 0·074 | 1·40 |
| 22222 | 3·54 | 40 | 0·147 | 3·36 |
| 44444 | 4·09 | 32 | 0·049 | −1·37 |
| 33342 | 4·16 | 31 | 0·043 | −1·36 |

**Table 5.** Correlation matrices for the data of Table 3

| | $d^2(e,g)$ | $R(e)$ | CV | Rating |
|---|---|---|---|---|
| (a) *Raw score correlation* | | | | |
| $d^2(e,g)$ | — | −1 | 0·983 | 0·824 |
| $R(e)$ | | — | −0·983 | −0·824 |
| CV | | | — | 0·746 |
| Rating | | | | — |
| (b) *Ranked score correlation* (*Spearman*) | | | | |
| $d^2(e,g)$ | — | −1 | 0·994 | 0·826 |
| $R(e)$ | | — | −0·994 | −0·826 |
| CV | | | — | 0·815 |
| Rating | | | | — |

(when the correct coding schema is used), and must be dismissed. Accordingly, to evaluate the contingency model, it suffices to contrast its predictions with those of the 'minimal prototype model'. This is done in Table 4. These results lead to the following correlation matrix [Table 5*a*: Raw scores correlation; Table 5*b*: Ranked scores correlation (Spearman)] As pointed out previously, no theoretical consideration justifies the use of transformed data in this evaluation. Hence conclusions based upon the analysis of the raw data are to be preferred. In any case, with both the raw scores and the ranked scores, the minimal model fares better than the contingency model. In particular, the advantage of the 'minimal prototype model' reaches significance for the untransformed data.

raw scores: $MS_{additional} = 7{\cdot}955$; $r^2_{additional} = 0{\cdot}246$, $F = 8{\cdot}72$, d.f. 1, 7, $P < 0{\cdot}05$;
ranked scores: $MS_{additional} = 1{\cdot}76$; $r^2_{additional} = 0{\cdot}021$, $F = 0{\cdot}47$, d.f. = 1, 7, n.s.

## Discussion

Clearly, when properly analysed, the experiment fails to support the claim of a superiority of the contingency model over the 'prototype models'. This all the more since the minimal prototype model would have fared better if specified, e.g. by allowing weighting of the variables and/or the levels within the variables; by using other distances than the Euclidean, etc. (in particular, due to the small number of exemplars of the test set, some weighted combinations of the features can fit the data perfectly). Nevertheless, the closeness of the predictions of the two models must be emphasized (as assessed by their correlation, cf. Table 4).

This correlation is not fortuitous at all. Contrary to the opinion of Richardson & Bhavnani, the contingency model is not *in its nature* different from a prototype model. Although their presentation did not make this point clear, the contingency model can easily be interpreted as a 'log-linear distance to the prototype' model (the logarithm of the contingency value gives an entropy which is a distance; see Appendix E). The CV model is simply *another member of the family* of the 'prototype plus distance to the prototype models'. As a consequence it cannot be opposed to this approach, though it could be proposed as a comfortable way of talking about a prototype model (for people reluctant about the distance formulation) or as a restatement of the links between current models of concept formation and information theory (cf. Attneave, 1959; Garner, 1962).

It must also be noted that all these models share some common weaknesses. First, they are not able to account for the effect of the contrasted categories (cf. Barsalou, 1982; Roth & Shoben, 1983), where the acquisition of a conceptual category depends on the intra-similarity of its exemplars as well as on inter-dissimilarity of its members to the other neighbouring categories. Second, they take as fixed the set of features describing the to-be-learned exemplars and ignore the problem of exactly what a feature is (the problem of defining clearly what a feature is in natural contexts is probably the main issue still to be resolved). Third, they ignore the relation between features (i.e. the possible existence of a 'featural grammar', cf. Sutherland, 1968, 1973; Tversky & Krantz, 1969; Reed, 1978; Davis & Christie, 1982). Actually the statement made by Richardson & Bhavnani (p. 517) can simply be reversed: 'the results imply...that the contingency model may have presented satisfactory results...simply because it emulates the prototype models' (in fact it is simply an 'entropic' variety of the distance to prototype models).

However, a point stressed by Richardson & Bhavnani should be taken in to account, namely the importance of the *structure* of the to-be-learned category. Although this point is hardly a new one (cf. among others: Attneave, 1957; Garner, 1962; Reed, 1972, 1978; Rosch & Mervis, 1975; Rosch, 1975, 1978, 1983; Homa *et al*., 1979; Barsalou, 1981; Medin *et al*., 1982; Homa, 1984; Malt & Smith, 1984), it is still important. But here, also the *a priori* structure imposed by the experimenter had to be separated from the structure effectively perceived by the subject, and especially with stimuli such as pseudo-faces where some components of the face could be of different importance, or could interact in a complex way, as acknowledged by the authors (cf. also Tversky & Krantz, 1969; Davies & Christie, 1982). To illustrate this point, compare (Figs 1a, b) the similarity matrix of, on the one hand, the 'objective' structure of the 'acquisition set' of Richardson & Bhavnani (where the resemblance between two faces is given by the number of common features, in agreement with the minimal prototype model), with, on the other hand, the similarity matrix given by 18 subjects sorting the same set of faces (where the resemblance between two faces is given by the number of subjects who sorted the faces together). At first glance, it is clear that the family resemblance is – on the whole – a good predictor of the sorting task results (correlation between the two matrices: 0·872; Stress 1 = 0·1094). More interesting, however, is the comparison of the internal structure of the categories, as it could be described for example by an additive-tree representation (cf. Sattah & Tversky, 1977; Abdi *et al*., 1984). Then, the differences between the internal structures of the categories appear more clearly. The subjects do not weight equally either the different variables or the levels within a variable. Their behaviour illustrates that the 'subjective'

(*a*)

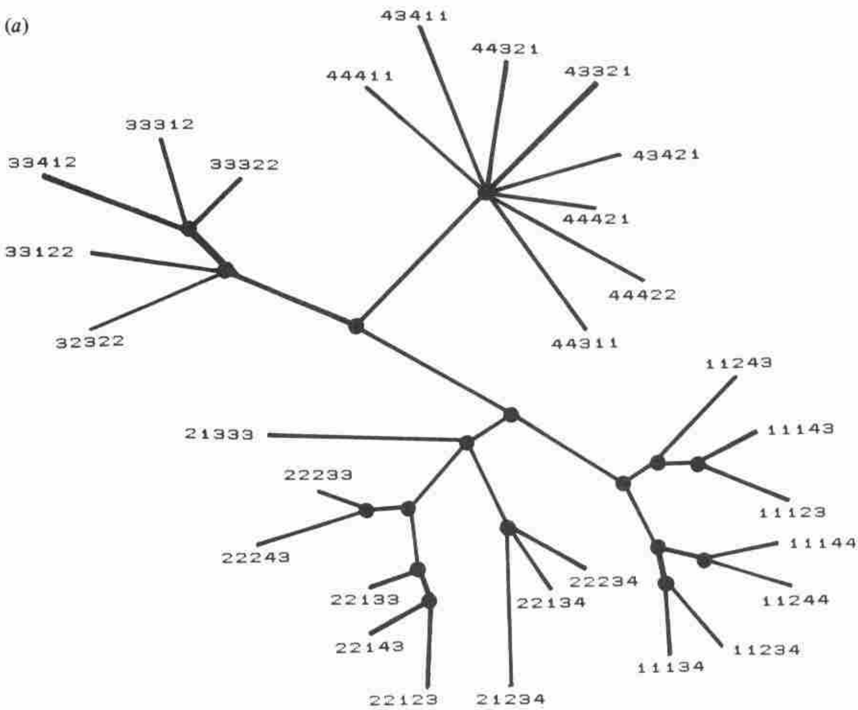


**Figure 1(a)**. Additive-tree representation of the learning-set of Richardson & Bhavnani. 'Objective resemblance': The resemblance of two faces is the number of features they agree on. ($r$ between the original matrix and the tree-distance = 0·88. Part of explained variance = 77·1 per cent, Stress 1 = 0·144).

composition or structure of a set of stimuli could be drastically different from those of the experimenter. Clearly the problem of a typology of the subjective organization of the conceptual categories and its effect on acquisition still remains.

To conclude, the alleged advantage of the contingency model is not supported by the experimental data. Moreover, the opposition between a contingency model and a distance model is not well-founded. However it can be suggested that the important issue in the study of concept formation is not the search for the *ultimate explicative model*, but instead to be able to find out when a particular model (if any) is adequate to describe what a particular subject does when learning a particular concept from, say, a set of exemplars with a particular internal organization embedded in a particular context. Accordingly, the 'minimal prototype model' is proposed as a very general device to test any new tentative model. By doing so, only the meaningful newcomers will be retained. Hopefully, this should ultimately help to reduce the current inflation of models for categories and concept formation.

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(*b*)

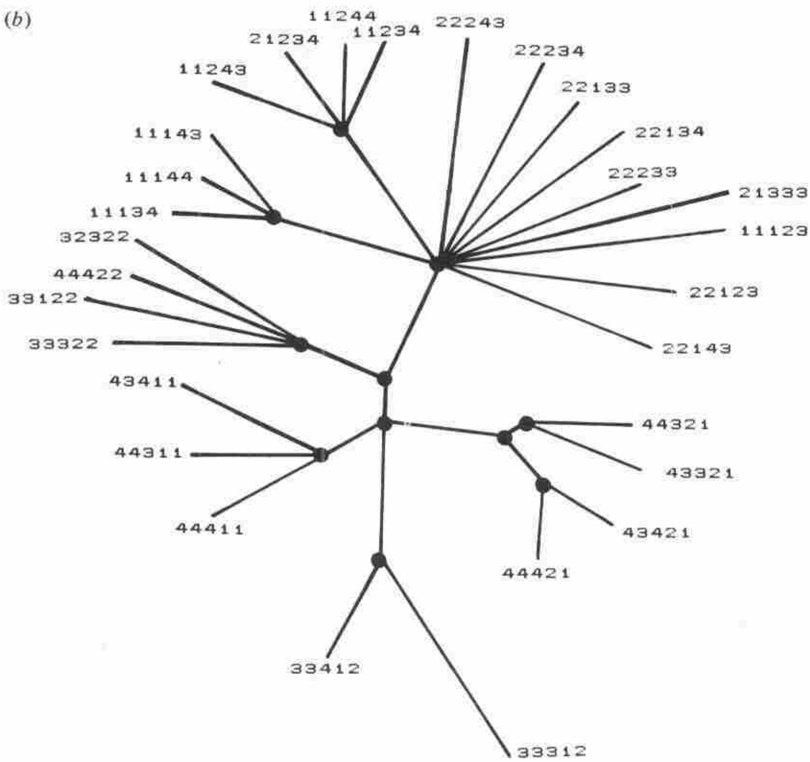


**Figure 1(b).** Additive-tree representation of the learning-set of Richardson & Bhavnani. 'Subjective resemblance': The resemblance of two faces is the number of subjects that sort together the faces. ($r$ between the original matrix and the tree-distance = 0·80. Part of explained variance = 63·8 per cent, Stress 1 = 0·119).

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## Appendix A

### Formal presentation of CV model

Define $V = \{v_1, \ldots, v_W\}$ a set of nominal variables, each $v_i$ being itself a set of features or levels. Thus, an exemplar $e$ of a concept is a $W$-uple of the form:

$$e = [l_1(e), \ldots, l_i(e), \ldots, l_W(e)],$$

where $l_i\,(e)$ is the level (or feature) of the $i$th nominal variable of the exemplar $e$ (recall that the levels of one nominal variable are exclusive). To each level of a nominal variable, we can associate a probability (with the constraint that the probabilities of the levels of one nominal variable sum to one). This probability will be estimated from the frequency of the level in the family of exemplars to be learned.

Then, *under the independence hypothesis*, the probability of observing any exemplar of the concept under scrutiny can easily be computed from the probability attached to each level of each nominal variable (the 'marginal probabilities'). Namely by using the (classical) formula:

$$\mathrm{CV}_e = P(e) = P[l_1(e), \ldots, l_i(e), \ldots, l_W(e)] = \prod_{i=1}^{W} p_i(e).$$

with $p_i\,(e)$: probability attached to the level of $i$th variable realized by exemplar $e$.

Richardson & Bhavnani called this probability the contingency value of exemplar $e$.

## Appendix B

### Formal presentation of the distance approach of the 'minimal prototype model'

Each exemplar is composed of a set of nominal variables, each of these being in turn composed of a set of mutually exclusive features or levels. Denote by $M$ the number of features or levels for all the variables (i.e. $M$ is the sum for all the variables of the number of levels). Then each possible exemplar could be represented by a vertex of a hypercube or (this amounts to the same thing), by a vector in an $M$-dimensional Euclidean space. That is to say, by a sequence of $M$ 1 or 0, where a 1 (resp. 0) means that the exemplar possesses (resp. does not possess) the feature corresponding to the digit.

Let $\mathbf{e}_k$ the vector standing for $e_k$ the $k$th exemplar of the to-be-learned set.

The set of exemplars can then be seen as a cloud of points in an $M$-dimensional Euclidean space. Call $g$ its centre of gravity whose coordinates are simply obtained by averaging: for each of its coordinates, sum the 1s and divide by $N$, that is:

$$g_i = \frac{1}{N}\sum_{k=1}^{N} e_{ik} = \frac{1}{N} e_{i\cdot}$$

with $e_{ik} = 1$ if exemplar $k$ possesses the $i$th feature, 0 if not. And $\mathbf{g} = (g_1, \ldots, g_i, \ldots, g_M) = \Sigma_k \, \mathbf{e}_k/N$, is the vector representing $g$.

The typicality of any new exemplar $f$ is assumed to be a function of its squared Euclidean distance to $g$.

Let $\mathbf{f} = (f_1, \ldots, f_i, \ldots, f_M)$ be the vector standing for $f$. Then the 'typicality' of $f$ is a function of:

$$d^2(f, g) = \sum_{i=1}^{M} (f_i - g_i)^2 = \| \mathbf{f} - \mathbf{g} \|^2.$$

## Appendix C

### Equivalence between the family resemblance approach and the distance approach

Define the resemblance between two exemplars by the number of features they share (i.e. by set-intersection).

Denote the resemblance of a new exemplar, say $f$, and $e_k$ by $r(f, e_k)$.

Denote by $R(f)$ the overall family resemblance between $f$ and the whole learning set of $N$ exemplars:

$$R(f) = \sum_{k=1}^{N} r(f, e_k).$$

Suffice to expand and rewrite this equation to see that $R(f)$ and $d(f, g)$ are closely related, namely that:

$$d^2(f, g) = W + \sum_{i=1}^{M} e_{i\cdot}^2 - \frac{2}{N}\sum_{i=1}^{M} g_i e_{i\cdot} = W + \frac{1}{N^2}\sum_{i=1}^{M} e_i^2 - \frac{2}{N} R(f).$$

Or:

$$R(f) = \left[ WN + \frac{1}{N}\sum_{i=1}^{M} e_{i\cdot}^2 - N d^2(f, g) \right] \Big/ 2.$$

Hence $R(f)$ is a monotonic function (precisely a quadratic function) of the distance to the centre of gravity. Note, incidentally, that the 'minimal prototype model' is obviously a particular case of the contrast model of Tversky (1977) when only the intersection between features term of this model is kept.

## Appendix D

### Equivalence between a 'distributed memory model' and the 'minimal prototype model'

The general formulation used here is adapted from Knapp & Anderson (1984). Assume that $\mathbf{e}_k$ denotes the column vector of order $M$ describing the $k$th exemplar. Denote by $N$ the number of exemplars.

Denote by $\mathbf{e}_k^T$ the transposed vector.

For convenience, assume $\mathbf{e}_k^T\mathbf{e}_k = 1$ (the vectors have a length of 1). $\mathbf{g}$ denote the vector with the coordinates of $g$ the barycentre. Recall two properties:

1. (cf. Coxon, 1982):

$$\mathbf{e}_k^T\,\mathbf{e}_{k'} = \cos(e_k e_{k'}) = 1 - d^2(e_k, e_{k'})/2.$$

with $d(e_k, e_{k'})$ being the usual Euclidean distance.

2. Huygens' theorem (cf. Benzécri, 1973; Greenacre, 1984)

$$\sum_k^N d^2(x, e_k) = Nd^2(x, g) + \Sigma\, d^2(e_k, g).$$

As there is just one category to-be-learned, the response associated with this category is always the same. This response is assumed to be a vector denoted $\mathbf{r}$ (a column vector of order $M$).

Learning in a distributed memory system is equivalent to the construction of the auto-associative matrix $\mathbf{A}$:

$$\mathbf{A} = \Sigma\mathbf{A}_k = \Sigma\mathbf{r}\mathbf{e}_k^T = \mathbf{r}\,\Sigma\,\mathbf{e}_k^T.$$

The output of the system to a new stimulus $f$ coded by vector $\mathbf{f}$ is given by:

$$\begin{aligned}\mathbf{Af} &= \mathbf{r}\,\Sigma\,\mathbf{e}_k^T\mathbf{f} = \mathbf{r}\,\Sigma\cos(e_k, f) = \mathbf{r}\,\Sigma\,[1 - d^2(e_k, f)/2]\\ &= N\mathbf{r} - \mathbf{r}\,\Sigma\,d^2(e_k, g) - \mathbf{r}Nd^2(f, g)/2\\ &= \mathbf{r}Nd^2(f, g)/2 + \text{constant}.\end{aligned}$$

Hence the response is a linear function of the squared Euclidean distance from $f$ to $g$. Thus, the 'minimal prototype model' is equivalent to a 'distributed memory model'.

## Appendix E

### The CV model is a 'prototype plus (log-linear) distance to prototype' model

Denote by $\mathrm{CV}_e = \prod_i p_i(e)$ the contingency value of exemplar $e$. Define:

$$d_l(e, e') = \sum_i |\log p_i(e) - \log p_i(e')|.$$

$d_l$ verifies the following properties and hence is a distance (1 and 2 follow directly from the definition, for 3 suffice to consider the possible cases and to check they are always in agreement with the inequality):

$$(1)\ d_l(e, e) = 0; \quad (2)\ d_l(e, e') \geqslant 0; \quad (3)\ d_l(e, e') + d_l(e', e'') \geqslant d_l(e, e'').$$

Now, consider the particular case of $m$ (the modal prototype). CV is maximal for $m$. Hence

$$\begin{aligned}d_l(e, m) &= \Sigma\,|\log p_i(e) - \log p_i(m)| = -\Sigma\log p_i(e) + \Sigma\log p_i(m)\\ &= -\log(\mathrm{CV}_e) + \log(\mathrm{CV}_{\max}) = -\log(\mathrm{CV}_e) + \text{constant}.\end{aligned}$$

Hence the predictions derived from the CVs are identical to the predictions derived from a (log-linear) distance to prototype model.