CONGRUENCE

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The *congruence* between two configurations of points quantifies their similarity. The configurations to be compared are, in general, produced by factor analytic methods (e.g., principal component analysis, correspondence analysis) that decompose an "observations by variables" data matrix and produce one set of factor scores for the observations and one set of factor scores (often called the *loadings*) for the variables. The congruence between two sets of factor scores collected on the same units (which can be observations or variables) measures the similarity between these two sets of scores. If, for example, two different types of factor analysis are performed on the same data set, the *congruence* between the two solutions is evaluated by the similarity of the configurations of the factor scores produced by these two techniques.

This entry presents three coefficients used to evaluate congruence. The first coefficient is called the *coefficient of congruence:* It measures the similarity of two configurations by computing a cosine between matrices of factor scores. The second and third coefficients are the R_V *coefficient* and the *Mantel coefficient.* These two coefficients evaluate the similarity of the *whole configuration* of units. In order to do so, the factor scores of the units are first transformed into a units-by-units square matrix, which reflects the configuration of similarity between the units; and then the similarity between the configurations is measured by a coefficient. For the R_V coefficient, the configuration between the units is obtained by computing a matrix of scalar products between the units, and a cosine between two scalar product matrices evaluates the similarity between two configurations. For the Mantel coefficient, the configuration between the units is obtained by computing a matrix of distance between the units, and a coefficient of correlation between two distance matrices evaluates the similarity between the two configurations described by the distance matrices.

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[*] **In Bruce Frey (Ed.), *The SAGE Encyclopedia of Research Design.* Thousand Oaks, CA: Sage. 2022**

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The *congruence coefficient* was first defined by Cyril Burt—under the name of *unadjusted correlation*—as a measure of the similarity between two factorial configurations. The name *congruence coefficient* was later tailored by Ledyard Tucker. The congruence coefficient is also sometimes called a *monotonicity coefficient.*

The R_V *coefficient* was introduced by Yves Escoufier as a measure of similarity between squared symmetric matrices (specifically: positive semidefinite matrices) and as a theoretical tool to analyze multivariate techniques. The R_V coefficient is used in several statistical techniques, such as STATIS and DISTATIS. In order to compare rectangular matrices with the R_V or the Mantel coefficients, the first step is to transform these rectangular matrices into square matrices.

The *Mantel coefficient* was originally introduced by Nathan Mantel in epidemiology but it is now widely used in ecology.

The congruence and the Mantel coefficients are cosines (recall that the coefficient of correlation is a *centered* cosine), and as such, they take values between –1 and +1. The R_V coefficient is also a cosine, but because it is a cosine between two matrices of scalar products (which, technically speaking, are *positive semidefinite* matrices), it corresponds actually to a *squared* cosine, and therefore the R_V coefficient takes values between 0 and 1.

The computational formulas of these three coefficients are almost identical, but their usage and theoretical foundations differ because these coefficients are applied to *different* types of matrices. Also, their sampling distributions differ because of the types of matrices on which they are applied.

Notations and Computational Formulas

A vector is an ordered list of numbers, it is denoted by a lower bold letter (e.g., **x**), its elements are denoted by the same letter typeset in italic with a subscript indicating the order of the element (e.g., x_1 or x_i). By default, vectors are written as columns of numbers: So, for example, the vector **x** with three elements can be written as:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} .$$

The transpose operation transforms a column vector into a row vector, it is denoted by the superscript $^\mathsf{T}$, so the previous column vector **x,** when transposed, becomes a row vector denoted $\mathbf{x}^\mathsf{T} = [x_1, x_2, x_3]$. Geometrically, vectors can be interpreted as points in a multidimensional space (with the dimension of the space being equal to the number of elements of the vector). In this framework, the *cosine* between two vectors, denoted, **x** and **y,** each with I elements is defined as:

$$\cos(\mathbf{x},\mathbf{y}) = \frac{\sum_{i}^{I} x_i \times y_i}{\sqrt{\left(\sum_{i}^{I} x_i^2\right) \times \left(\sum_{i}^{I} y_i^2\right)}} .$$

Matrices are array of numbers whose dimensions correspond to their number of rows and columns. Matrices are denoted by upper cases bold letter and their elements are denoted by italic lower case letters with subscript indicating their row and column. So, for example, **X** can denote an I rows by J column matrix whose generic element is denoted $x_{i,j}$. The "vec" operation transforms a matrix into a column vector whose entries are the elements of the matrix (i.e., it unfolds the matrix). The cosine between two matrices is defined as the vector cosine between the "vectorized" version of these two matrices. The "trace" operation applies to square matrices and gives the sum of the diagonal elements of this matrix. The transpose operation of a matrix is denoted by the superscript $^{\mathsf{T}}$, it exchanges the roles of the rows and the columns of a matrix, for example, if **X** is an I by J, $\mathbf{X}^{\mathsf{T}}$ is a J by I matrix. When two matrices are written next to each other, this indicates matrix multiplication.

Congruence Coefficient

The congruence coefficient is defined when both matrices have the same number of rows and columns (i.e., $J = K$). These matrices can store factor scores (for observations) or factor loadings (for variables). The congruence coefficient is denoted φ or sometimes r_c, and it can be computed with three different equivalent formulas:

$$\varphi = r_c = \frac{\sum_{i,j} x_{i,j} y_{i,j}}{\sqrt{\left(\sum_{i,j} x_{i,j}^2\right)\left(\sum_{i,j} y_{i,j}^2\right)}} \qquad (1)$$

$$= \frac{\text{vec}\{\mathbf{X}\}^{\mathsf{T}} \text{vec}\{\mathbf{Y}\}}{\sqrt{\left(\text{vec}\{\mathbf{X}\}^{\mathsf{T}} \text{vec}\{\mathbf{X}\}\right)\left(\text{vec}\{\mathbf{Y}\}^{\mathsf{T}} \text{vec}\{\mathbf{Y}\}\right)}} \qquad (2)$$

$$= \frac{\text{trace}\{\mathbf{XY}^{\mathsf{T}}\}}{\sqrt{\left(\text{trace}\{\mathbf{XX}^{\mathsf{T}}\}\right)\left(\text{trace}\{\mathbf{YY}^{\mathsf{T}}\}\right)}} \qquad (3)$$

R_V *Coefficient*

The R_V coefficient was defined by Robert Escoufier as a similarity coefficient between positive semidefinite matrices (e.g., matrices such as correlation, covariance, or cross-product matrices). Pierre Robert and Robert Escoufier later pointed out that the R_V coefficient had important mathematical properties because most multivariate analysis techniques amount to maximizing—with suitable constraints—this coefficient. Recall, at this point, that a matrix **S** is called *positive semidefinite* when it can be obtained as the product of a matrix by its transpose. Formally, we say that **S** is positive semidefinite when there exists a matrix **X** such that

$$\mathbf{S} = \mathbf{XX}^{\mathsf{T}}. \qquad (4)$$

Note that as a consequence of the definition, positive semidefinite matrices are square and symmetric, and that their diagonal elements are always larger than or equal to zero.

If **S** and **T** denote two positive semidefinite matrices of same dimensions, the R_V coefficient between them is defined as

$$R_V = \frac{\text{trace}\{\mathbf{S}^{\mathsf{T}}\mathbf{T}\}}{\sqrt{\left(\text{trace}\{\mathbf{S}^{\mathsf{T}}\mathbf{S}\}\right) \times \left(\text{trace}\{\mathbf{T}^{\mathsf{T}}\mathbf{T}\}\right)}} \qquad (5)$$

This formula is computationally equivalent to

$$R_V = \frac{\text{vec}\{\mathbf{S}\}^{\mathsf{T}}\text{vec}\{\mathbf{T}\}}{\sqrt{\left(\text{vec}\{\mathbf{S}\}^{\mathsf{T}}\text{vec}\{\mathbf{S}\}\right)\left(\text{vec}\{\mathbf{T}\}^{\mathsf{T}}\text{vec}\{\mathbf{T}\}\right)}} \qquad (6)$$

$$= \frac{\sum_i^I \sum_i^I s_{i,j} t_{i,j}}{\sqrt{\left(\sum_i^I \sum_i^I s_{i,j}^2\right)\left(\sum_i^I \sum_i^I t_{i,j}^2\right)}} \qquad (7)$$

For rectangular matrices, the first step is to transform the matrices into positive semi-definite matrices by multiplying each matrix by its transpose. So, in order to compute the value of the R_V coefficient between the I by J matrix **X** and the I by K matrix **Y**, the first step it to compute the cross-product matrices **S** and **T**

$$\mathbf{S} = \mathbf{X}\mathbf{X}^{\top} \quad \text{and} \quad \mathbf{T} = \mathbf{Y}\mathbf{Y}^{\top} \tag{8}$$

If we combine Equations 5 and 8, we find that the R_V coefficient between the two rectangular matrices **X** and **Y** is equal to

$$R_V = \frac{\operatorname{trace}\left\{\mathbf{X}\mathbf{X}^{\top}\mathbf{Y}\mathbf{Y}^{\top}\right\}}{\sqrt{\operatorname{trace}\left\{\mathbf{X}\mathbf{X}^{\top}\mathbf{X}\mathbf{X}^{\top}\right\} \times \operatorname{trace}\left\{\mathbf{Y}\mathbf{Y}^{\top}\mathbf{Y}\mathbf{Y}^{\top}\right\}}} \tag{9}$$

The comparison of Equations 3 and 9 shows that the congruence and the R_V coefficients are equivalent only in the case of positive semidefinite matrices.

From a linear algebra point of view, the numerator of the R_V coefficient corresponds to a scalar product between positive semidefinite matrices and therefore gives to this set of matrices the structure of a vector space. Within this framework, the denominator of the R_V coefficient is called the *Frobenius* or *Schur* or *Hilbert-Schmidt* matrix scalar product, and the R_V coefficient is a *cosine* between matrices. This vector space structure is responsible for the mathematical properties of the R_V coefficient.

Mantel Coefficient

For the Mantel coefficient, if the data are not already in the form of distances, then the first step is to transform these data into distances. These distances can be Euclidean distances, but any other type of distance will work. If **D** and **B** denote the two I by I distance matrices of interest (with respective generic elements $d_{i,j}$ and $b_{i,j}$), then the Mantel coefficient between these two matrices is denoted r_M, and it is computed as the coefficient of correlation between their off-diagonal elements as

$$r_M = \frac{\sum_{i=1}^{I-1}\sum_{j=i+1}^{I}(d_{i,j}-\bar{d})(b_{i,j}-\bar{b})}{\sqrt{\left[\sum_{i=1}^{I-1}\sum_{j=i+1}^{I}(d_{i,j}-\bar{d})^2\right]\left[\sum_{i=1}^{I-1}\sum_{j=i+1}^{I}(b_{i,j}-\bar{b})^2\right]}} \tag{10}$$

(where $\bar{d}$ and $\bar{b}$ are the mean of the off-diagonal elements of, respectively, matrices **D** and **B**).

Tests and Sampling Distributions

The congruence, the R_V, and the Mantel coefficients all quantify the similarity between two matrices. An obvious practical problem is to be able to perform statistical testing on the value of a given coefficient. In particular it is often important to be able to decide whether a value of coefficient could have been obtained by chance alone. To perform such statistical tests, one needs to derive the sampling distribution of these coefficients under the null hypothesis (i.e., in order to test whether the population coefficient is null). More sophisticated testing requires one to derive the sampling distribution for different values of the population parameters. So far, analytical methods have failed to completely characterize such distributions, but computational approaches have been used with some success. Because the congruence, the R_V , and the Mantel coefficients are used with different types of matrices, their sampling distributions differ, and so work done with each type of coefficient has been carried independently of the others.

Some approximations for the sampling distributions have been derived recently for the congruence coefficient and the R_V coefficient, with particular attention given to the R_V coefficient. The sampling distribution for the Mantel coefficient has not been satisfactorily approximated, and the statistical tests provided for this coefficient rely mostly on permutation tests.

Congruence Coefficient

Recognizing that analytical methods were unsuccessful, Bruce Korth and Ledyard Tucker decided to use Monte Carlo simulations to gain some insights into the sampling distribution of the congruence coefficient. Their work was completed by Wendy Broadbooks and Patricia Elmore. From this work, it seems that the sampling distribution of the congruence coefficient depends on several parameters, including the original factorial structure and the intensity of the population coefficient, and therefore no simple picture emerges, but some approximations can be used. In particular, for testing that a congruence coefficient is null in the population, an approximate conservative test is to use Fisher's Z transform and to treat the congruence coefficient like a coefficient of correlation. Broadbooks and Elmore have provided tables for population values different from zero. With the availability of fast computers, these tables can easily be extended to accommodate specific cases.

Example

Here we use an example from Hervé Abdi and Dominique Valentin (2007). Two wine experts are rating six wines on three different scales. The results of their ratings are provided in the two matrices below, denoted **X** and **Y**:

$$\mathbf{X} = \begin{bmatrix} 1 & 6 & 7 \\ 5 & 3 & 2 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \\ 2 & 5 & 4 \\ 3 & 4 & 4 \end{bmatrix} \text{ and } \mathbf{Y} = \begin{bmatrix} 3 & 6 & 7 \\ 4 & 4 & 3 \\ 7 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 6 & 6 \\ 1 & 7 & 5 \end{bmatrix} . \tag{11}$$

For computing the congruence coefficient, these two matrices are transformed into two vectors of 6 (products) × 3 (variables) = 18 elements each, and a cosine (cf. Equation 1) is computed between these two vectors. This gives a value of the coefficient of congruence of $\varphi = .7381$. In order to evaluate whether this value is significantly different from zero, a permutation test with 10,000 permutations was performed. In this test, the *rows* of one of the matrices were randomly permuted, and the coefficient of congruence was computed for each of these 10,000 permutations. The probability of obtaining a value of $\varphi = .7381$ under the null hypothesis was evaluated as the proportion of the congruence coefficients larger than $\varphi = .7381$. This gave a value of $p = .0259$, which is small enough to reject the null hypothesis at the .05 α-level, and thus one can conclude that the agreement between the ratings of these two experts cannot be attributed to chance.

R_V Coefficient

Statistical approaches for the R_V coefficient have focused on permutation tests. In this framework, the permutations are performed on the entries of each column of the rectangular matrices **X** and **Y** used to create the matrices **S** and **T** or directly on the rows *and* columns of **S** and **T**. Interestingly by Frédérique Kazi-Aoual and colleagues have shown that the mean and the variance of the permutation test distribution can be approximated directly from **S** and **T**. To so, the first step is to derive an index of the dimensionality or rank of the matrices. This index, denoted β_S (for matrix $\mathbf{S} = \mathbf{X}\mathbf{X}^T$), is also known as ν in the brain imaging literature, where it is called a *sphericity* index and is used as an estimation of the number of degrees of freedom for multivariate tests of the general linear model. The index β_S depends on the set of the L eigenvalues of the **S** matrix (i.e., **S** has rank L), denoted ${}_S\lambda_\ell$, and it is defined as

$$\beta_{\mathbf{S}} = \frac{\left(\sum_{\ell}^{L} {}_{\mathbf{S}}\lambda\ell\right)^2}{\sum_{\ell}^{L} {}_{\mathbf{S}}\lambda_{\ell}^2} = \frac{\operatorname{trace}\{\mathbf{S}\}^2}{\operatorname{trace}\{\mathbf{SS}\}} . \qquad (12)$$

The mean of the set of permutated coefficients between matrices **S** and **T** is then equal to

$$E(R_V) = \frac{\sqrt{\beta_{\mathbf{S}}\beta_{\mathbf{T}}}}{I-1} . \qquad (13)$$

The case of the variance is more complex and involves computing three preliminary quantities for each matrix. The first quantity denoted $\delta_{\mathbf{S}}$ is (for matrix **S**) equal to

$$\delta_{\mathbf{S}} = \frac{\sum_{i}^{I} s_{i,i}^2}{\sum_{\ell}^{L} {}_{\mathbf{S}}\lambda_{\ell}^2} . \qquad (14)$$

The second one is denoted $\alpha_{\mathbf{S}}$ for matrix $\mathbf{S}$ and is defined as

$$\alpha_{\mathbf{S}} = I - 1 - \beta_{\mathbf{S}} . \qquad (15)$$

The third one is denoted $C_{\mathbf{S}}$ (for matrix **S**) and is defined as

$$C_{\mathbf{S}} = \frac{(I-1)\left[I(I+1)\delta_{\mathbf{S}} - (I-1)(\beta_{\mathbf{S}}+2)\right]}{\alpha_{\mathbf{S}}(I-3)} . \qquad (16)$$

With these notations, the variance of the permuted coefficients is obtained as

$$V(R_V) = \alpha_{\mathbf{S}}\alpha_{\mathbf{T}} \times \frac{2I(I-1)+(I-3)C_{\mathbf{S}}C_{\mathbf{T}}}{I(I+1)(I-2)(I-1)^3} . \qquad (17)$$

For very large matrices, the sampling distribution of the permutated coefficients is relatively similar to a normal distribution (even though it is, in general, *not* normal), and therefore one can use a *Z* criterion to perform null hypothesis testing or to compute confidence intervals. For example, the criterion

$$Z_{R_V} = \frac{R_V - E(R_V)}{\sqrt{V(R_V)}} \qquad (18)$$

can be used to test the null hypothesis that the observed value of R_V was due to chance.

The problem of the lack of normality of the permutation-based sampling distribution of the R_V coefficient has been addressed by Moonseong Heo and Ruben Gabriel, who have suggested "normalizing" the sampling distribution by using a log transformation. Recently Julie Josse, Jerome Pagès, and François Husson have refined this approach and indicated that a gamma distribution would give an even better approximation.

R_V Coefficient: An Example

As an example, of the computation of the R_V we use the two scalar product matrices obtained from the matrices used to illustrate the congruence coefficient (cf. Equation 11). For the present example, these original matrices are centered (i.e., the mean of each column has been subtracted from each element of the column) prior to computing the scalar product matrices. Specifically, if $\overline{\mathbf{X}}$ and $\overline{\mathbf{Y}}$ denote the centered matrices derived from **X** and **Y**, we obtain the following scalar product matrices:

$$\mathbf{S} = \overline{\mathbf{X}}\,\overline{\mathbf{X}}^{\mathsf{T}} = \begin{bmatrix} 29.56 & -8.78 & -20.78 & -20.11 & 12.89 & 7.22 \\ -8.78 & 2.89 & 5.89 & 5.56 & -3.44 & -2.11 \\ -20.78 & 5.89 & 14.89 & 14.56 & -9.44 & -5.11 \\ -20.11 & 5.56 & 14.56 & 16.22 & -10.78 & -5.44 \\ 12.89 & -3.44 & -9.44 & -10.78 & 7.22 & 3.56 \\ 7.22 & -2.11 & -5.11 & -5.44 & 3.56 & 1.89 \end{bmatrix} \tag{19}$$

and

$$\mathbf{T} = \overline{\mathbf{Y}}\,\overline{\mathbf{Y}}^{\mathsf{T}} = \begin{bmatrix} 11.81 & -3.69 & -15.19 & -9.69 & 8.97 & 7.81 \\ -3.69 & 1.81 & 7.31 & 1.81 & -3.53 & -3.69 \\ -15.19 & 7.31 & 34.81 & 9.31 & -16.03 & -20.19 \\ -9.69 & 1.81 & 9.31 & 10.81 & -6.53 & -5.69 \\ 8.97 & -3.53 & -16.03 & -6.53 & 8.14 & 8.97 \\ 7.81 & -3.69 & -20.19 & -5.69 & 8.97 & 12.81 \end{bmatrix}. \tag{20}$$

We find the following value for the R_V coefficient:

$$\begin{aligned} R_V &= \frac{\sum_i^I \sum_j^I s_{i,j} t_{i,j}}{\sqrt{\left(\sum_i^I \sum_j^I s_{i,j}^2\right)\left(\sum_i^I \sum_j^I t_{i,j}^2\right)}} \\ &= \frac{(29.56\times 11.81)+(-8.78\times -3.69)+\cdots+(1.89\times 12.81)}{\sqrt{\left[(29.56)+(-8.78)^2+\cdots+(1.89)^2\right]\left[(11.81)^2+(-3.69)^2+\cdots+(12.81)^2\right]}} \\ &= .7936\,. \end{aligned} \tag{21}$$

To test the significance of a value of $R_V = .7936$, we first compute the following quantities:

$$\begin{aligned} \beta_{\mathbf{S}} &= 1.0954 \quad & \alpha_{\mathbf{S}} &= 3.9046 \\ \delta_{\mathbf{S}} &= 0.2951 \quad & C_{\mathbf{S}} &- 1.3162 \\ \beta_{\mathbf{T}} &= 1.3851 \quad & \alpha_{\mathbf{T}} &= 3.6149 \\ \delta_{\mathbf{T}} &= 0.3666 \quad & C_{\mathbf{T}} &= -0.7045 \end{aligned} \tag{22}$$

Plugging these values into Equations 13, 17, and 18, we find

$$E(R_V) = 0.2464,$$
$$V(R_V) = 0.0422, \text{ and} \quad (23)$$
$$Z_{R_V} = 2.66.$$

Assuming a normal distribution for the Z_{R_V} gives a p value of .0077, which would allow for the rejection of the null hypothesis for the observed value of the R_V coefficient.

R_V Coefficient: Permutation Test

As an alternative approach to evaluate whether the value of $R_V = .7936$ is significantly different from zero, a permutation test with 10,000 permutations was performed. In this test, the whole set of rows *and* columns (i.e., the *same* permutation of I elements is used to permute rows and columns) of one of the scalar product matrices was randomly permuted, and the R_V coefficient was computed for each of these 10,000 permutations. The probability of obtaining a value of $R_V = .7936$ under the null hypothesis was evaluated as the proportion of the R_V coefficients larger than $R_V = .7936$. This gave a value of $p = .0281$, which is small enough to reject the null hypothesis at the .05 alpha level. It is worth noting that the normal approximation gives a more liberal (i.e., smaller) value of p than does the nonparametric permutation test (which is more accurate in this case because the sampling distribution of R_V is not normal).

Mantel Coefficient

The exact sampling distribution of the Mantel coefficient is not known. Numerical simulations suggest that, when the distance matrices originate from different independent populations, the sampling distribution of the Mantel coefficient is symmetric (though not normal) with a zero mean. In fact, Mantel, in his original paper, presented some approximations for the variance of the sampling distributions of r_M (derived from the permutation test) and suggested that a normal approximation could be used, but the problem is still open. In practice, though, the probability associated to a specific value of r_M is derived from permutation tests.

Example

As an example, two distance matrices derived from the congruence coefficient example (cf. Equation 11) are used. These distance matrices can be computed directly from the scalar product matrices used to illustrate the computation of the R_V coefficient (cf. Equations 19 and

20). Specifically, if $\mathbf{S}$ is a scalar product matrix and if $\mathbf{s}$ denotes the vector containing the diagonal elements of $\mathbf{S}$, and if $\mathbf{1}$ denotes an I by 1 vector of ones, then the matrix $\mathbf{D}$ of the squared Euclidean distances between the elements of $\mathbf{S}$ is obtained as (cf. Equation 4):

$$\mathbf{D} = \mathbf{1}\mathbf{s}^{\top} + \mathbf{s}\mathbf{1}^{\top} - 2\mathbf{S} \,. \tag{24}$$

Using Equation 24, we transform the scalar-product matrices from Equations 19 and 20 into the following distance matrices:

$$\mathbf{D} = \begin{bmatrix} 0 & 50 & 86 & 86 & 11 & 17 \\ 50 & 0 & 6 & 8 & 17 & 9 \\ 86 & 6 & 0 & 2 & 41 & 27 \\ 86 & 8 & 2 & 0 & 45 & 29 \\ 11 & 17 & 41 & 45 & 0 & 2 \\ 17 & 9 & 27 & 29 & 2 & 0 \end{bmatrix} \tag{25}$$

and

$$\mathbf{T} = \begin{bmatrix} 0 & 21 & 77 & 42 & 2 & 9 \\ 21 & 0 & 22 & 9 & 17 & 22 \\ 77 & 22 & 0 & 27 & 75 & 88 \\ 42 & 9 & 27 & 0 & 32 & 35 \\ 2 & 17 & 75 & 32 & 0 & 3 \\ 9 & 22 & 88 & 35 & 3 & 0 \end{bmatrix} \,. \tag{26}$$

For computing the Mantel coefficient, the upper diagonal elements of each of these two matrices are stored into a vector of $\frac{1}{2} I \times (I-1) = 15$ elements, and the standard coefficient of correlation is computed between these two vectors. This gives a value of the Mantel coefficient of r_M = .5769. In order to evaluate whether this value is significantly different from zero, a permutation test with 10,000 permutations was performed. In this test, the whole set of rows *and* columns (i.e., the *same* permutation of I elements is used to permute rows and columns) of one of the matrices was randomly permuted, and the Mantel coefficient was computed for each of these 10,000 permutations. The probability of obtaining a value of r_M = .5769. under the null hypothesis was evaluated as the proportion of the Mantel coefficients larger than r_M = .5769. This gave a value of p = .0265, which is small enough to reject the null hypothesis at the .05 alpha level.

Conclusion

The congruence, R_V, and Mantel coefficients all measure slightly different aspects of the notion of congruence. The congruence coefficient is sensitive to the pattern of similarity of the columns of the matrices and therefore will not detect similar configurations when one of the configurations is rotated or dilated. By contrast, both the R_V coefficient and the Mantel coefficients are sensitive to the whole configuration and are insensitive to changes in configuration that involve rotation or dilatation. The R_V coefficient has the additional merit of being theoretically linked to most multivariate methods and of being the base of Procrustes methods such as STATIS or DISTATIS.

Hervé Abdi

See also Coefficients of Correlation; Alienation, and Determination; Principal Components Analysis; R^2; Sampling Distributions; Matrix Algebra.

Further Readings

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