Probability of Error Analysis of 4-QAM OFDM Systems with Random Residual Frequency Offset

P. C. Weeraddana†, R. M. A. P. Rajatheva† and Hlaing Minn††
Telecommunications Field of Study, Asian Institute of Technology†, P.O.Box 4, Klong Luang 12120, Pathumthani, Thailand, Email: rajath@ait.ac.th, Department of Electrical Engineering, University of Texas at Dallas,†† P.O.Box 830688, EC 33, Richardson, TX 75083-0688, U.S.A, Email: hlaing.minn@utdallas.edu

***Abstract*— In this paper, we derive closed-form symbol error rate (SER) expressions for orthogonal frequency division multiplexing (OFDM) systems with residual carrier frequency offset (CFO). We treat CFO/residual CFO as a random parameter in this study. In particular, we consider channel-independent as well as channel-dependent random residual CFOs. We derive SER expressions for 4-quadrature amplitude modulation (4-QAM) OFDM systems in the cases of additive white Gaussian noise (AWGN) and frequency flat Rayleigh fading channels. The simulation results are provided to verify the accuracy of the new SER expressions.**

I. Introduction

In the performance analysis of OFDM systems, one approach is to treat inrer carrier interference (ICI) as a Gaussian process based on the central limit theorem [1] which does not yield satisfactory results at high signal to noise ratios (SNR) [2]. In contrast, the approach followed in [3] uses the characteristic function and Beaulieu series to derive exact BER expressions for AWGN channel in the presence of ICI where the probability of error is expressed conditioned on the normalized frequency offset. In [4], the authors have derived exact BER/SER expressions for AWGN, frequency-flat and frequency-selective channels with fixed CFO error. Recently, [5] analyzed error performance of BPSK OFDM systems with a uniform CFO which is assumed to be independent of the channel. Generally, after CFO estimation and compensation, the residual CFO becomes channel-dependent. Hence, error performance of OFDM systems with channel-dependent random residual CFO is of practical interest.

In this paper we present an approach for SER analysis of 4-QAM OFDM systems with a random (residual) CFO. Procedures discussed in [6] for M-QAM are adapted accordingly for our derivations discussed here. We consider both scenarios where the (residual) CFO is independent of the channel and dependent on the channel. The technical contents and the structure of this paper are as follows. In Section II we present the system model. In Section III we derive closed-form SER expressions for AWGN and frequency-flat Rayleigh fading channels with a channel-independent uniformly-distributed (residual) CFO. Section IV addresses the channel-dependent CFO case where we obtain the SER expression for a frequency-flat Rayleigh fading channel with a random residual CFO which, conditioned on the channel, is Gaussian-distributed. The SER analyses in Section III and IV assume that the perfect channel knowledge is available at the receiver. Section V provides simulation results to verify our theoretical results. Maximum likelihood CFO estimators are included in our simulation to evaluate the applicability of our analytical results to practical systems. Finally, Section VI concludes the paper.

II. System Model and Analysis

We assume quasi-static frequency selective fading channels. We use the following notations. $(.)^H$ and $(.)^T$ denote the Hermitian transpose and transpose operations respectively. Further $|z|$, $\angle z$, $\Re(z)$ and $\Im(z)$ denote the absolute value, angle, real and imaginary components of the complex quantity z, respectively. In the presence of normalized (by the sub-carrier spacing) CFO v, the received signal vector $\mathbf{r}$ is given by [7]

$$\boldsymbol{r} = \boldsymbol{\Gamma}(v)\boldsymbol{S}\boldsymbol{h} + \boldsymbol{w} = \sqrt{N}\boldsymbol{\Gamma}(v)\boldsymbol{F}^H\boldsymbol{H}\boldsymbol{c} + \boldsymbol{w} \tag{1}$$

where $\boldsymbol{r} = [r_0\ r_1\ \cdots\ r_{N-1}]^T$, $\boldsymbol{c} = [c_0\ c_1\ \cdots\ c_{N-1}]^T$, $\boldsymbol{h} = [h_0\ h_1\ \cdots\ h_{L-1}]^T$, $\boldsymbol{w} = [w_0\ w_1\ \cdots\ w_{N-1}]^T$, $\boldsymbol{\Gamma}(v) = \mathrm{diag}[1\ e^{j2\pi v/N}\ \ldots\ e^{j2\pi(N-1)v/N}]$ and $\boldsymbol{H} = \mathrm{diag}\{\boldsymbol{F_L}\boldsymbol{h}\} = \mathrm{diag}[H_0\ H_1\ \cdots\ H_{N-1}]^T$. The N-point unitary discrete Fourier transform (DFT) matrix is denoted by $\boldsymbol{F} = [\boldsymbol{f}_0\boldsymbol{f}_1\cdots\boldsymbol{f}_{N-1}]$ where $\boldsymbol{f}_k = [1, e^{-j2\pi k/N}, \cdots, e^{-j2\pi(N-1)/N}]^T/\sqrt{N}$. We define $\boldsymbol{F}_L = [\boldsymbol{f}_0\boldsymbol{f}_1\cdots\boldsymbol{f}_{L-1}]$. Here $\{h_n\}$ denote the channel impulse response (CIR) coefficients and L is the number of CIR taps. $\{w_n\}$ are independent and identically-distributed (i.i.d.) zero-mean circularly-symmetric complex Gaussian noise samples each having a variance of σ^2 per dimension. $\{c_n\}$ are independent equi-probable frequency domain transmit symbols and the corresponding time-domain signal vector is given by $\boldsymbol{s}{=}[s_0\ s_1\ \cdots\ s_{N-1}]^T = \boldsymbol{F}^H\boldsymbol{c}$. The time-domain signal matrix in (1) is defined by $[\boldsymbol{S}]_{k,n}{=}s_{k-n}$, $0 \le k \le N-1$, $0 \le n \le L-1$ with $s_k{=}\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} c_n e^{j2\pi nk/N}$ for $k = L-1, \cdots, N-1$.

Let $\hat{v}$ be the estimated frequency offset, where $\hat{v}{=}v{+}v_\Delta$. We denote the residual CFO as v_Δ. So we can write the received symbol for the kth sub-carrier with the transmitted symbol c_k as [5]

$$R_k = \sqrt{N}c_kH_kI'_0 + \sqrt{N}\sum_{l=0,l\neq k}^{N-1} c_lH_lI'_{l-k} + n'_k \qquad k = 0, 1, \ldots, N-1 \tag{2}$$

where $\{n'_k\}$ are i.i.d. random variables having the same statistical properties as $\{w_n\}$ and

$$I'_{l-k} \quad \approx \quad \begin{cases} \frac{\pi v_\Delta}{N}[-\cot(\frac{\pi(l-k)}{N})+j] & \text{if } l \neq k \\ 1-j\pi\frac{N-1}{N}v_\Delta & \text{if } l=k. \end{cases} \tag{3}$$

III. Performance Analysis with channel-independent residual CFO/CFO

For the channel-independent (residual) CFO case, SER on a particular (say 0th) sub-carier of the ith OFDM symbol conditioned on the other $N-1$ sub-carrier symbols is obtained by solving the following

$$P_s(\xi|\boldsymbol{a}_i) = \int\int P_s(\xi|v_\Delta, \boldsymbol{h}, \boldsymbol{a}_i)\, f_v(v_\Delta) f(\boldsymbol{h}) dv_\Delta d\boldsymbol{h} \tag{4}$$

where $f_v(v_\Delta)$ and $f(\boldsymbol{h})$ are pdfs of residual CFO/CFO and channel respectively, and $P_s(\xi|v_\Delta, \boldsymbol{h}, \boldsymbol{a}_i)$ represents the SER conditioned on v_Δ, $\boldsymbol{h}$ and $\boldsymbol{a}_i$. Here $\boldsymbol{a}_i = [c_{1,i} \;\; c_{2,i} \;\; \cdots \;\; c_{N-1,i}]^T$. In the following sections A and B, we consider the uniformly distributed CFO, while in the section C we address a Gaussian-distributed CFO. A square M-QAM modulation can be considered as a combination of two quadrature (say I and Q) $\sqrt{M}$-PAM (pulse amplitude modulaion) schemes, each with half the total power. Since a correct QAM decision is made only when a correct decision is independently made on each of these PAM modulations, then the symbol error probability for a square QAM can be expressed as [6]

$$\begin{aligned} P_s(\text{error})|_{M-QAM,E_s} &= P_{I\sqrt{M},\frac{E_s}{2}} + P_{Q\sqrt{M},\frac{E_s}{2}} \\ &- P_{I\sqrt{M},\frac{E_s}{2}} \times P_{Q\sqrt{M},\frac{E_s}{2}}, \end{aligned} \tag{5}$$

where $P_{IM,E_s} = [P_s(\text{error})|_{M-PAM,E_s}]_I$ and $P_Q = [P_s(error)|_{M-PAM,E_s}]_Q$. This holds true even for the case when the symbol error probability is conditioned on some random parameters. Writing the equation (2) with some slight modifications to the symbols c_k's, we can express the received symbol on the kth sub-carrier for M-PAM OFDM as

$$\begin{aligned} R_k &= \sqrt{\epsilon_s N} A_k H_k I'_0 + \sqrt{\epsilon_s N}\sum_{l=0,l\neq k}^{N-1} A_l H_l I'_{l-k} \\ &+ n'_k; \qquad\qquad\qquad k=0,1,\ldots,N-1, \end{aligned} \tag{6}$$

where $\epsilon_s = \frac{3E_s}{M^2-1}$, $A_m \in \{-(M-1)..-1,1..(M-1)\}$ and E_s is the symbol energy. Now consider the M-QAM OFDM signal with the signal points $c_k = c_{I_k} + jc_{Q_k}$ with $c_{I_k}, c_{Q_k} \in \{-(M-1)..-1,1..(M-1)\}$. Thus we can write the equivalent two quadrature components of M-QAM signal on the I and Q-axis of the complex plane for the zeroth sub-carrier as

$$\begin{aligned} Y_I &= \sqrt{\epsilon_s}|\alpha_0|\Re(c_0 I'_0) + \sqrt{\epsilon_s N}\sum_{l=1}^{N-1}\Re(\zeta c_l H_l I'_l) + n_I \\ Y_Q &= \sqrt{\epsilon_s}|\alpha_0|\Im(c_0 I'_0) + \sqrt{\epsilon_s N}\sum_{l=1}^{N-1}\Im(\zeta c_l H_l I'_l) + n_Q, \end{aligned} \tag{7}$$

where $Y_I = \Re(R_0)$, $Y_Q = \Im(R_0)$, $\alpha_0 = \sqrt{N}H_0$, $\zeta = e^{-j\angle\alpha_0}$ and n_I, n_Q are i.i.d. real Gaussian random variables with zero mean and variance σ^2.

A. AWGN Channel with Uniformly Distributed CFO

For the AWGN channel, (7) reduces to

$$\begin{aligned} Y_I &= \sqrt{\epsilon_s}\Re(c_0 I'_0) + \sqrt{\epsilon_s}\sum_{l=1}^{N-1}\Re(c_l I'_l) + n_I \\ Y_Q &= \sqrt{\epsilon_s}\Im(c_0 I'_0) + \sqrt{\epsilon_s}\sum_{l=1}^{N-1}\Im(c_l I'_l) + n_Q. \end{aligned} \tag{8}$$

For an M-QAM OFDM system with $M=4$ and a particular symbol c_0^* on the zero-th sub-carrier, we have $P_{IM,E_s}|\boldsymbol{a}_i, v_\Delta, c_0^* = P_{IM,E_s}|\boldsymbol{a}_i, v_\Delta$ and $P_{QM,E_s}|\boldsymbol{a}_i, v_\Delta, c_0^* = P_{QM,E_s}|\boldsymbol{a}_i, v_\Delta$. Then we can derive $P_{IM,E_s}|\boldsymbol{a}_i, v_\Delta$ and $P_{QM,E_s}|\boldsymbol{a}_i, v_\Delta$ using (8) as follows [6, eq.(8.3)]:

$$\begin{aligned} P_{IM,E_s}|\boldsymbol{a}_i, v_\Delta &= \tfrac{M-1}{M}Q\left(\sqrt{\epsilon_s}\tfrac{[\Re(c_0^* I'_0) - \frac{\pi v_\Delta}{N}\Re(X_i)]}{\sigma}\right) \\ &+ \tfrac{M-1}{M}Q\left(\sqrt{\epsilon_s}\tfrac{[\Re(c_0^* I'_0) + \frac{\pi v_\Delta}{N}\Re(X_i)]}{\sigma}\right). \end{aligned} \tag{9}$$

$$\begin{aligned} P_{QM,E_s}|\boldsymbol{a}_i, v_\Delta &= \tfrac{M-1}{M}Q\left(\sqrt{\epsilon_s}\tfrac{[\Im(c_0^* I'_0) - \frac{\pi v_\Delta}{N}\Im(X_i)]}{\sigma}\right) \\ &+ \tfrac{M-1}{M}Q\left(\sqrt{\epsilon_s}\tfrac{[\Im(c_0^* I'_0) + \frac{\pi v_\Delta}{N}\Im(X_i)]}{\sigma}\right). \end{aligned} \tag{10}$$

Here $X_i = \sum_{l=1}^{N-1} c_{l,i}[-\cot(\frac{\pi l}{N}) + j]$. Without loss of generality, for the 4-QAM case c_0^* is taken to be equal to $(1+j)$. Using (5), (9) and (10), we can derive the conditional SER for 4-QAM OFDM as given in (11) where $\alpha_{Ii} = \pi\left[\frac{N-1}{N} - \frac{\Re(X_i)}{N}\right]$, $\beta_{Ii} = \pi\left[\frac{N-1}{N} + \frac{\Re(X_i)}{N}\right]$, $\alpha_{Qi} = \pi\left[\frac{N-1}{N} - \frac{\Im(X_i)}{N}\right]$, $\beta_{Qi} = \pi\left[\frac{N-1}{N} + \frac{\Im(X_i)}{N}\right]$ and $2\gamma = \frac{2E_b}{N_0} = \frac{E_s}{N_0}$. E_b and E_s represent bit energy and symbol energy, respectively, and the complex noise variance is denoted by $N_0 = 2\sigma^2$. Now we define $I_1(\mu,\lambda) = \int Q(\mu + \lambda v_\Delta) f_v(v_\Delta) dv_\Delta$, $I_2(\mu,\lambda_1,\lambda_2) = \int Q(\mu+\lambda_1 v_\Delta) Q(\mu+\lambda_2 v_\Delta) f_v(v_\Delta) dv_\Delta$ and $I_3(\mu,\lambda,\omega_1,\omega_2) = \int_{\omega_1}^{\omega_2} Q(\mu+\lambda x) e^{-x^2/2} dx$ where $f_v(v_\Delta)$ is considered to be a uniform distribution over $[-b,b]$ and μ is non-zero. Then we can derive

$$I_1(\mu,\lambda) = \begin{cases} Q(\mu) & \text{if } \lambda = 0 \\ \frac{1}{2b\lambda}\Big[(\mu+\lambda x)Q(\mu+\lambda x) & \\ \quad -\frac{1}{\sqrt{2\pi}}e^{-\frac{(\mu+\lambda x)^2}{2}}\Big]_{-b}^{b} & \text{if } \lambda \neq 0 \end{cases} \tag{12}$$

$$I_2(\mu,\lambda_1,\lambda_1) = \begin{cases} \big[g_{I_2}(x,\mu,\lambda_1,\lambda_2)\big]_{-b}^{b} & \\ +\mathcal{I}_3 & \text{if } \lambda_1,\lambda_2 \neq 0 \\ I_1(\mu,\lambda_1).I_1(\mu,\lambda_2) & \text{else} \end{cases} \tag{13}$$

where $\mathcal{I}_3 = \frac{\mu(\lambda_1-\lambda_2)}{\sqrt{8\pi}\lambda_1\lambda_2 b} I_3\left(\frac{\mu(\lambda_1-\lambda_2)}{\lambda_1}, \frac{\lambda_2}{\lambda_1}, \mu-\lambda_1 b, \mu+\lambda_1 b\right)$, $\big[g(x)\big]_{-b}^{b} = g(b) - g(-b)$ and g is any arbitrary function

$$\begin{aligned} P_s(\xi|\boldsymbol{a}_i, v_\Delta) &= \tfrac{1}{2}Q\left(\sqrt{2\gamma}[1+\alpha_{Ii}v_\Delta]\right) + \tfrac{1}{2}Q\left(\sqrt{2\gamma}[1+\beta_{Ii}v_\Delta]\right) + \tfrac{1}{2}Q\left(\sqrt{2\gamma}[1-\alpha_{Qi}v_\Delta]\right) + \tfrac{1}{2}Q\left(\sqrt{2\gamma}[1-\beta_{Qi}v_\Delta]\right) \\ &-\tfrac{1}{4}Q\left(\sqrt{2\gamma}[1+\alpha_{Ii}v_\Delta]\right)\cdot Q\left(\sqrt{2\gamma}[1-\alpha_{Qi}v_\Delta]\right) - \tfrac{1}{4}Q\left(\sqrt{2\gamma}[1+\alpha_{Ii}v_\Delta]\right)\cdot Q\left(\sqrt{2\gamma}[1-\beta_{Qi}v_\Delta]\right) \\ &-\tfrac{1}{4}Q\left(\sqrt{2\gamma}[1+\beta_{Ii}v_\Delta]\right)\cdot Q\left(\sqrt{2\gamma}[1-\alpha_{Qi}v_\Delta]\right) - \tfrac{1}{4}Q\left(\sqrt{2\gamma}[1+\beta_{Ii}v_\Delta]\right)\cdot Q\left(\sqrt{2\gamma}[1-\beta_{Qi}v_\Delta]\right) \end{aligned} \tag{11}$$

defined over $[-b,b]$. $g_{I_2}(x,\mu,\lambda_1,\lambda_2)$ is given by (14). Using (12) and (13), we can obtain the SER conditioned on $\boldsymbol{a}_i$ as given in (15). Averaging over all $\boldsymbol{a}_i$ combinations leads to the SER

$$P_s(\xi) = \frac{1}{2^{2(N-1)}} \sum_i P_s(\xi|\boldsymbol{a}_i), \tag{16}$$

where $\sum_i \equiv \sum_{c_1\in\mathcal{A}} \sum_{c_2\in\mathcal{A}} \dots \sum_{c_{N-1}\in\mathcal{A}}$ and $\mathcal{A} = \{1+j, 1-j, -1+j, -1-j\}$.

B. Frequency-Flat Rayleigh Fading Channel with Uniformly Distributed CFO

When the frequency flat Raleigh fading is concerned, the equivalent quadrature components in (7) reduce to

$$\begin{aligned} Y_I &= \sqrt{\epsilon_s}|\alpha_0|\Re(c_0 I_0') + \sqrt{\epsilon_s}|\alpha_0| \textstyle\sum_{l=0}^{N-1} \Re(\zeta c_l I_l') + n_I \\ Y_Q &= \sqrt{\epsilon_s}|\alpha_0|\Im(c_0 I_0') + \sqrt{\epsilon_s}|\alpha_0| \textstyle\sum_{l=0}^{N-1} \Im(\zeta c_l I_l') + n_Q. \end{aligned} \tag{17}$$

Following the same set of arguments we can easily derive the conditional SER, $P_s(\xi|\boldsymbol{a}_i, v_\Delta, |\alpha_0|)$, replacing $\sqrt{2\gamma}$ in (11) with $\sqrt{2\gamma}|\alpha_0|$. Here we use the distribution of $|\alpha_0|$, $f_{\alpha_0}(|\alpha_0|) = \frac{|\alpha_0|}{\sigma_R^2} \exp\big(-\frac{|\alpha_0|^2}{2\sigma_R^2}\big)$ where α_0 is a zero-mean complex Gaussian random variable with a variance of σ_R^2 per dimension. Now we define the integrals $T_1(\beta) = \int_{-b}^{b}\int_0^\infty Q\big(a(\beta,v_\Delta)|\alpha_0|\big) f_{\alpha_0}(|\alpha_0|) f_v(v_\Delta) d|\alpha_0| dv_\Delta$ and $T_2(\alpha,\beta,v_\Delta) = \int_0^\infty Q\big(a(\alpha,v_\Delta)|\alpha_0|\big) Q\big(a(\beta,v_\Delta)|\alpha_0|\big) \times f_{\alpha_0}(|\alpha_0|) d|\alpha_0|$ where $a(\beta,v_\Delta) = \sqrt{2\gamma}(1+\beta v_\Delta)$. Then we can solve the above integrations to obtain

$$T_1(\beta) = \begin{cases} \frac{1}{2} - \frac{\sqrt{2\gamma}\sigma_R}{2\sqrt{1+2\gamma\sigma_R^2}} & \text{if } \beta = 0 \\ \frac{1}{2} - \frac{\left[\sqrt{1+2\gamma\sigma_R^2 x^2}\right]_{1-\beta b}^{1+\beta b}}{4\sqrt{2\gamma}\beta\sigma_R b} & \text{if } \beta \neq 0 \end{cases} \tag{18}$$

and $T_2(\alpha,\beta,v_\Delta)$ is given by (19) with the function $m(\alpha,v_\Delta) = \frac{\sigma_R}{\sqrt{1+\sigma_R^2 a^2(\alpha,v_\Delta)}}$. Now we want to find $T_2(\alpha,\beta)$ which is given by $T_2(\alpha,\beta) = \int_{-b}^{b} T_2(\alpha,\beta,v_\Delta) f_v(v_\Delta) dv_\Delta$. For notational simplicity, we denote $T_2(\alpha,\beta)$ as

$$T_2(\alpha,\beta) = 0.25 - T_{2'}(\alpha) - T_{2'}(\beta) + T_{2'}(\alpha,\beta) + T_{2'}(\beta,\alpha) \tag{20}$$

where $T_{2'}(\alpha) = \int_{-b}^{b} \frac{a(\alpha,v_\Delta)m(\alpha,v_\Delta)}{8b} dv_\Delta$, $T_{2'}(\alpha,\beta) = \int_{-b}^{b} \frac{a(\alpha,v_\Delta)m(\alpha,v_\Delta)}{4\pi b} \cot^{-1}\big[\frac{1}{a(\beta,v_\Delta)m(\alpha,v_\Delta)}\big] dv_\Delta$ and $f_v(v_\Delta) = \frac{1}{2b} : v_\Delta \in [-b,b]$. After some mathematical manipulations it can be easily shown that $T_{2'}(\alpha) = \frac{1}{4} - \frac{T_1(\alpha)}{2}$ and

$$T_{2'}(\alpha,\beta) = \begin{cases} \big[g_{1T_{2'}}(x,\alpha,\beta)\big]_{1-\alpha b}^{1+\alpha b} & \text{if } \alpha \neq 0 \\ \big[g_{2T_{2'}}(x,\alpha,\beta)\big]_{-b}^{b} & \text{if } \alpha = 0, \beta \neq 0 \\ \frac{1}{2\pi\rho}\cot^{-1}(\rho) & \text{if } \alpha = 0, \beta = 0 \end{cases} \tag{21}$$

where $g_{1T_{2'}}(x,\alpha,\beta)$ and $g_{1T_{2'}}(x,\alpha,\beta)$ are defined in (22) and (23), respectively, $\eta = \frac{\beta}{\alpha}$, $\bar{\gamma} = \sqrt{2\gamma}\sigma_R$, $\rho = \frac{\sqrt{1+\bar{\gamma}^2}}{\bar{\gamma}}$, $Q = \frac{\eta(1-\eta)}{1+\eta^2}$ and $R = \frac{\sqrt{1+\eta^2+\bar{\gamma}^2(1-\eta)^2}}{\bar{\gamma}(1+\eta^2)}$. Now we have derived the expressions for $T_1(\beta)$ and $T_2(\alpha,\beta)$, and using (18) and (20) we can easily write the SER conditioned on $\boldsymbol{a}_i$, $P_s(\xi|\boldsymbol{a}_i)$, as given in (24). Averaging over all $\boldsymbol{a}_i$ combinations leads to the SER which is given in (16).

C. AWGN and Frequency-Flat Rayleigh Fading Channels with Perfect Power Control

To evaluate (4), we should know the pdf of v_Δ. As far as maximum likelihood (ML) estimators are concerned, we can observe the nature of the pdf of v_Δ conditioned on the channel. Asymptotic properties of the maximum likelihood estimate (MLE) indicate that if the regularity conditions are satisfied, then the MLE of the unknown parameter $\boldsymbol{\theta}$ is asymptotically Gaussian-distributed as $\hat{\boldsymbol{\theta}} \sim \mathcal{N}(\boldsymbol{\theta}, \boldsymbol{I}^{-1}(\boldsymbol{\theta}))$ where $\boldsymbol{I}(\boldsymbol{\theta})$ is

$$\begin{aligned} g_{I_2}(x,\mu,\lambda_1,\lambda_2) = {}& \frac{1}{2b\lambda_2}\big(\mu+\lambda_2 x\big)Q\big(\mu+\lambda_2 x\big)Q\big(\mu+\lambda_1 x\big) - \frac{1}{\sqrt{8\pi}\lambda_2 b}Q\big(\mu+\lambda_1 x\big)\exp\left(\frac{-(\mu+\lambda_2 x)^2}{2}\right) \\ & - \frac{1}{\sqrt{8\pi}\lambda_1 b}Q\big(\mu+\lambda_2 x\big)\exp\left(\frac{-(\mu+\lambda_2 x)^2}{2}\right) + \frac{(\lambda_1+\lambda_2)}{\sqrt{8\pi(\lambda_1^2+\lambda_2^2)}\lambda_1\lambda_2 b}\exp\left(\frac{-\mu^2(\lambda_1-\lambda_2)^2}{2(\lambda_1^2+\lambda_2^2)}\right)Q\left(\frac{\mu(\lambda_1+\lambda_2)}{\sqrt{\lambda_1^2+\lambda_2^2}} + \sqrt{\lambda_1^2+\lambda_2^2}\,x\right). \end{aligned} \tag{14}$$

$$\begin{aligned} P_s(\xi|\boldsymbol{a}_i) = {}& \tfrac{1}{2}I_1(\sqrt{2\gamma},\sqrt{2\gamma}\alpha_{Ii}) + \tfrac{1}{2}I_1(\sqrt{2\gamma},\sqrt{2\gamma}\beta_{Ii}) + \tfrac{1}{2}I_1(\sqrt{2\gamma},-\sqrt{2\gamma}\alpha_{Qi}) + \tfrac{1}{2}I_1(\sqrt{2\gamma},-\sqrt{2\gamma}\beta_{Qi}) \\ & - \tfrac{1}{4}I_2(\sqrt{2\gamma},\sqrt{2\gamma}\alpha_{Ii},-\sqrt{2\gamma}\alpha_{Qi}) - \tfrac{1}{4}I_2(\sqrt{2\gamma},\sqrt{2\gamma}\alpha_{Ii},-\sqrt{2\gamma}\beta_{Qi}) - \tfrac{1}{4}I_2(\sqrt{2\gamma},\sqrt{2\gamma}\beta_{Ii},-\sqrt{2\gamma}\alpha_{Qi}) \\ & - \tfrac{1}{4}I_2(\sqrt{2\gamma},\sqrt{2\gamma}\beta_{Ii},-\sqrt{2\gamma}\beta_{Qi}). \end{aligned} \tag{15}$$

$$T_2(\alpha,\beta,v_\Delta) = \frac{1}{4} - \frac{a(\alpha,v_\Delta)m(\alpha,v_\Delta)}{2\pi}\left(\frac{\pi}{2} - \cot^{-1}\left[\frac{1}{a(\beta,v_\Delta)m(\alpha,v_\Delta)}\right]\right) - \frac{a(\beta,v_\Delta)m(\beta,v_\Delta)}{2\pi}\left(\frac{\pi}{2} - \cot^{-1}\left[\frac{1}{a(\alpha,v_\Delta)m(\beta,v_\Delta)}\right]\right) \tag{19}$$

$$\begin{aligned} g_{1T_{2'}}(x,\alpha,\beta) = {}& \frac{1}{4\pi\alpha b\bar{\gamma}}\sqrt{1+\bar{\gamma}^2x^2}\cot^{-1}\left[\frac{\sqrt{1+\bar{\gamma}^2x^2}}{\bar{\gamma}(\eta x+1-\eta)}\right] - \frac{\eta\sqrt{1+\eta^2+\bar{\gamma}^2(1-\eta)^2}}{4\pi\alpha b\bar{\gamma}(1+\eta^2)}\tan^{-1}\left[\frac{x+Q}{R}\right] \\ & + \frac{1-\eta}{4\pi\alpha b(1+\eta^2)}\ln\left[\sqrt{\bar{\gamma}^2(1+\eta^2)x^2 + 2\eta\bar{\gamma}^2(1-\eta)x + \bar{\gamma}^2(1-\eta)^2 + 1}\right] \end{aligned} \tag{22}$$

$$g_{2T_{2'}}(x,\alpha,\beta) = \frac{1}{4\pi\rho b}\left(x\cot^{-1}\left[\frac{\rho}{1+\beta x}\right] - \frac{\rho}{\beta}\ln\left[\sqrt{(1+\beta x)^2+\rho^2}\right] + \frac{1}{\beta}\tan^{-1}\left[\frac{1+\beta x}{\rho}\right]\right) \tag{23}$$

$$\begin{aligned} P_s(\xi|\boldsymbol{a}_i) = {}& \tfrac{1}{2}T_1(\alpha_{Ii}) + \tfrac{1}{2}T_1(\beta_{Ii}) + \tfrac{1}{2}T_1(-\alpha_{Qi}) + \tfrac{1}{2}T_1(-\beta_{Qi}) - \tfrac{1}{4}T_2(\alpha_{Ii},-\alpha_{Qi}) - \tfrac{1}{4}T_2(\alpha_{Ii},-\beta_{Qi}) \\ & - \tfrac{1}{4}T_2(\beta_{Ii},-\alpha_{Qi}) - \tfrac{1}{4}T_2(\beta_{Ii},-\beta_{Qi}) \end{aligned} \tag{24}$$

the Fisher information matrix evaluated at the true value of the unknown parameter [8]. Hence, it is reasonable to use the conditional pdf of v_Δ as $f(v_\Delta|\boldsymbol{h}) = \mathcal{N}(0, \boldsymbol{I}^{-1}(\boldsymbol{\theta})) = \mathcal{N}(0, CRB|_{\boldsymbol{h}})$ where $CRB|_{\boldsymbol{h}}$ is the Cramer-Rao lower bound conditioned on the CIR. If we assume perfect power control, we can say $\boldsymbol{h}^H\boldsymbol{S}^H\boldsymbol{S}\boldsymbol{h}$ is constant and hence for a receiver with a CFO estimator, the pdf of the residual CFO can be considered as a Gaussian pdf independent of the channel resulting simply $f(v_\Delta|\boldsymbol{h}) = f(v_\Delta)$. If we consider arbitrary training signal samples $\{s_k\}$, the CRB for v derived for the ML joint estimation of v and $\boldsymbol{h}$ is given by [9] $CRB|_{\boldsymbol{h}} = \frac{N^2\sigma^2}{4\pi^2\boldsymbol{h}^H\boldsymbol{S}^H\boldsymbol{\Lambda}(\boldsymbol{I}_N-\boldsymbol{B})\boldsymbol{\Lambda}\boldsymbol{S}\boldsymbol{h}}$ where $\boldsymbol{B}{=}\boldsymbol{S}(\boldsymbol{S}^H\boldsymbol{S})^{-1}\boldsymbol{S}^H$ and $\boldsymbol{\Lambda} = \text{diag}\{0, 1, \ldots, N-1\}$. We use this $CRB|_{\boldsymbol{h}}$ in our subsequent derivations for flat fading Raleigh fading channel.

1) AWGN Channel: For the AWGN channel, (4) simply reduces to a single integral evaluation and the signal model for AWGN channel can be obtained from (1) as, $\boldsymbol{r} = \boldsymbol{\Gamma}(\boldsymbol{v})\boldsymbol{s} + \boldsymbol{w}$ where $\boldsymbol{s} = [s_0 s_1 \ldots s_{N-1}]^T$ is the training signal vector. The CRB of the CFO estimation for the aforementioned signal model is given by [10], $CRB = \frac{N^2\sigma^2}{4\pi^2\boldsymbol{s}^H\boldsymbol{\Lambda}^2\boldsymbol{s}} = \Omega$. Hence using the definition of $I_1(\mu, \lambda)$ we can easily show that $I_1(\mu, \lambda) = Q(\frac{\mu}{1+\Omega\lambda^2})$ as $f_v(v_\Delta) = \mathcal{N}(0, \Omega)$, [11, eq 3.66] and

$$\begin{aligned} I_2(\mu, \lambda_1, \lambda_2) \quad &= \tfrac{1}{2\pi}\int_0^{\frac{\pi}{2}-\phi_1} \exp\left(\frac{-\mu^2}{2{b_1}^2\sin^2\phi}\right)d\phi \\ &+ \tfrac{1}{2\pi}\int_0^{\frac{\pi}{2}-\phi_2} \exp\left(\frac{-\mu^2}{2{b_2}^2\sin^2\phi}\right)d\phi. \end{aligned} \tag{25}$$

$I_2(\mu, \lambda_1, \lambda_2)$ cannot be evaluated in closed-form and it shows similarities to the well known Craig's formula. For simplicity, define $\lambda_{1\Omega} = \sqrt{\Omega}\lambda_1$ and $\lambda_{2\Omega} = \sqrt{\Omega}\lambda_2$. So that $b_1 = \sqrt{\lambda_{1\Omega}^2 + 1}$, $b_2 = \sqrt{\lambda_{2\Omega}^2 + 1}$, $\phi_1 = \tan^{-1}(a_1 b_1)$, $\phi_2 = \tan^{-1}(a_2 b_2)$, $a_1 = \frac{\lambda_{1\Omega}^2 - \lambda_{1\Omega}\lambda_{2\Omega}+1}{\sqrt{(\lambda_{1\Omega}^2+1)(\lambda_{1\Omega}^2+\lambda_{2\Omega}^2+1)}}$ and $a_2 = \frac{\lambda_{2\Omega}^2 - \lambda_{1\Omega}\lambda_{2\Omega}+1}{\sqrt{(\lambda_{2\Omega}^2+1)(\lambda_{1\Omega}^2+\lambda_{2\Omega}^2+1)}}$. Then with some mathematical manipulations we obtain the SER conditioned on $\boldsymbol{a}_i$, $P_s(\xi|\boldsymbol{a}_i)$ as given in (15). Averaging over all $\boldsymbol{a}_i$ combinations gives the SER which is given by (16).

2) Frequency-Flat Rayleigh Fading Channel: When the frequency-flat fading channel is considered, the $CRB|_{\boldsymbol{h}}$ which was mentioned previously can be reduced to $CRB|_{\alpha_0} = \frac{2N^2}{\left(8\pi^2\boldsymbol{s}^H\boldsymbol{\Lambda}(\boldsymbol{I}_N-\boldsymbol{B})\boldsymbol{\Lambda}\boldsymbol{s}\right)}\frac{\sigma^2}{|\alpha_0|^2} {=} \frac{\Lambda}{|\alpha_0|^2}$ where α_0 is a zero-mean complex Gaussian random variable with a variance of σ_R^2 per dimension. Under the perfect power control, we can consider that $|\alpha_0|^2$ is constant while fixing $\boldsymbol{s}$. Here Λ was introduced for simplicity. Thus we have the pdf of residual CFO $f(v_\Delta|\alpha_0) = f_v(v_\Delta) = \mathcal{N}(0, \Lambda)$. So that using the conditional SER $P_s(\xi|\boldsymbol{a}_i, v_\Delta, |\alpha_0|)$ derived in section B, $CRB|_{\alpha_0}$, aforementioned $f_v(v_\Delta)$ and (4), we can derive the SER, following almost the same set of arguments which were used in the derivation of SER in Section C-1. The following parameter changes should be noticed carefully: $I_1(\mu, \lambda) = Q(\frac{\mu}{1+\Lambda\lambda^2})$ and parameters in $I_2(\mu, \lambda_1, \lambda_2)$; $b_1 = \sqrt{\lambda_{1\Lambda}^2 + 1}$, $b_2 = \sqrt{\lambda_{2\Lambda}^2 + 1}$, $\phi_1 = \tan^{-1}(a_1 b_1)$, $\phi_2 = \tan^{-1}(a_2 b_2)$, $a_1 = \frac{\lambda_{1\Lambda}^2 - \lambda_{1\Lambda}\lambda_{2\Lambda}+1}{\sqrt{(\lambda_{1\Lambda}^2+1)(\lambda_{1\Lambda}^2+\lambda_{2\Lambda}^2+1)}}$, $a_2 = \frac{\lambda_{2\Lambda}^2 - \lambda_{1\Lambda}\lambda_{2\Lambda}+1}{\sqrt{(\lambda_{2\Lambda}^2+1)(\lambda_{1\Lambda}^2+\lambda_{2\Lambda}^2+1)}}$ where $\lambda_{1\Lambda} = \sqrt{\Lambda}\lambda_1$ and $\lambda_{2\Lambda} = \sqrt{\Lambda}\lambda_2$. Hence the SER and the corresponding conditional SER are given by (16) and (15) respectively.

IV. Performance Analysis with Channel-Dependent Residual CFO

For the channel-dependent residual CFO scenario, the symbol error probability can be expressed as

$$P_s(\xi) = \int\int P_s(\xi|v_\Delta, \boldsymbol{h})\, f_v(v_\Delta|\boldsymbol{h}) f(\boldsymbol{h}) dv_\Delta d\boldsymbol{h}. \tag{26}$$

The closed-form solution to (26) for the frequency-flat Rayleigh fading channel is presented in the following. However, solving the above problem for the frequency-selective case appears to be intractable. The variance of the conditional Gaussian random variable $v_\Delta|\alpha_0$ for the frequency-flat Rayleigh fading channel is given by $CRB|_{\alpha_0} = \frac{\Lambda}{|\alpha_0|^2}$ for the MLE estimator [9] as mentioned before. Then averaging the conditional SER $P_s(\xi|\boldsymbol{a}_i, v_\Delta, |\alpha_0|)$ using $f_v(v_\Delta|\boldsymbol{h})$ we can obtain $P_s(\xi|\boldsymbol{a}_i, |\alpha_0|)$ which is given in (27) where $E\{.\}$ is the statistical expectation with respect to the random variable $X = \frac{v_\Delta|\alpha_0|}{\sqrt{\Lambda}}$ and $X \sim \mathcal{N}(0, 1)$. It is obvious that by observing the functions $I_1(\mu, \lambda)$ and $I_2(\mu, \lambda_1, \lambda_2)$ in Section C-2, we can write

$$\begin{aligned} P_s(\xi|\boldsymbol{a}_i, |\alpha_0|) \quad &= \tfrac{1}{2}I_1(\sqrt{2\gamma}|\alpha_0|, \sqrt{2\gamma}\alpha_{Ii}) + \ldots. \\ &\ldots. - \tfrac{1}{4}I_2(\sqrt{2\gamma}|\alpha_0|, \sqrt{2\gamma}\beta_{Ii}, -\sqrt{2\gamma}\beta_{Qi}). \end{aligned} \tag{28}$$

Next, after integrating $P_s(\xi|\boldsymbol{a}_i, |\alpha_0|)$ with $f_{\alpha_0}(|\alpha_0|)$ to remove the dependency of $|\alpha_0|$, we obtain the conditional SER $P_s(\xi|\boldsymbol{a}_i)$ as given in (29). We will define now $I_1^*(t_0, t_1) {=} \int_0^\infty I_1(t_0|\alpha_0|, t_1) f_{\alpha_0}(|\alpha_0|) d|\alpha_0|$ and $I_2^*(t_0, t_1, t_2) {=} \int_0^\infty I_2(t_0|\alpha_0|, t_1, t_2) f_{\alpha_0}(|\alpha_0|) d|\alpha_0|$. It can be shown that $I_1^*(t_0, t_1) = [1/2 - t_0\sigma_R/\sqrt{1 + t_0^2\sigma_R^2 + \Lambda t_1^2}]$ and with, $\varepsilon_1 = \frac{t_0\sigma_R}{b_1}$, $\varepsilon_2 = \frac{t_0\sigma_R}{b_2}$, $\psi_1 = \tan^{-1}(a_1 b_1)$, $\psi_2 = \tan^{-1}(a_2 b_2)$, $b_1 = \sqrt{t_{1\Lambda}^2 + 1}$, $b_2 = \sqrt{t_{2\Lambda}^2 + 1}$, $a_1 = \frac{t_{1\Lambda}^2 - t_{1\Lambda}t_{2\Lambda}+1}{\sqrt{(t_{1\Lambda}^2+1)(t_{1\Lambda}^2+t_{2\Lambda}^2+1)}}$, $a_2 = \frac{t_{2\Lambda}^2 - t_{1\Lambda}t_{2\Lambda}+1}{\sqrt{(t_{2\Lambda}^2+1)(t_{1\Lambda}^2+t_{2\Lambda}^2+1)}}$ where $t_{1\Lambda}$

$$\begin{aligned} P_s(\xi|\boldsymbol{a}_i, |\alpha_0|) \quad &= \tfrac{1}{2}E\Big\{Q\left(\sqrt{2\gamma}\big[|\alpha_0| + \sqrt{\Lambda}\alpha_{Ii}X\big]\right)\Big\} + \tfrac{1}{2}E\Big\{Q\left(\sqrt{2\gamma}\big[|\alpha_0| + \sqrt{\Lambda}\beta_{Ii}X\big]\right)\Big\}\ldots\ldots.. \\ &\ldots\ldots.. - \tfrac{1}{4}E\Big\{Q\left(\sqrt{2\gamma}\big[|\alpha_0| + \sqrt{\Lambda}\beta_{Ii}X\big]\right)Q\left(\sqrt{2\gamma}\big[|\alpha_0| - \sqrt{\Lambda}\beta_{Qi}X\big]\right)\Big\} \end{aligned} \tag{27}$$

$$\begin{aligned} P_s(\xi|\boldsymbol{a}_i) \quad &= \tfrac{1}{2}I_1^*(\sqrt{2\gamma}, \sqrt{2\gamma}\alpha_{Ii}) + \tfrac{1}{2}I_1^*(\sqrt{2\gamma}, \sqrt{2\gamma}\beta_{Ii}) + \tfrac{1}{2}I_1^*(\sqrt{2\gamma}, -\sqrt{2\gamma}\alpha_{Qi}) + \tfrac{1}{2}I_1^*(\sqrt{2\gamma}, -\sqrt{2\gamma}\beta_{Qi}) \\ &-\tfrac{1}{4}I_2^*(\sqrt{2\gamma}, \sqrt{2\gamma}\alpha_{Ii}, -\sqrt{2\gamma}\alpha_{Qi}) - \tfrac{1}{4}I_2^*(\sqrt{2\gamma}, \sqrt{2\gamma}\alpha_{Ii}, -\sqrt{2\gamma}\beta_{Qi}) - \tfrac{1}{4}I_2^*(\sqrt{2\gamma}, \sqrt{2\gamma}\beta_{Ii}, -\sqrt{2\gamma}\alpha_{Qi}) \\ &-\tfrac{1}{4}I_2^*(\sqrt{2\gamma}, \sqrt{2\gamma}\beta_{Ii}, -\sqrt{2\gamma}\beta_{Qi}). \end{aligned} \tag{29}$$

$= \sqrt{\Lambda}t_1$ and $t_{2\Lambda} = \sqrt{\Lambda}t_2$, $I_2^*(t_0, t_1, t_2)$ given in (30). Substituting the conditional SER in (29) into (16) will give the corresponding SER.

V. Simulation results and discussion

SER curves when the normalized (residual) CFO is uniformly distributed over $[-b, b]$ with $b = 0.05$ and $b = 0.1$ are shown in Fig. 1 for both AWGN and frequency-flat Rayleigh fading channels. The simulation results for $b = 0.05$ case match well with those calculated in (15)and (16), but there is a slight discrepancy for $b = 0.1$ case especially at low SER values. This discrepancy is simply due to the fact that the small (residual) CFO assumption in the analytical development is not closely matched by the uniform (residual) CFO with $b = 0.1$, and at these low SER values CFO has a more dominant effect on SER than the noise does. Note that at high SNR, residual CFO would typically be quite small and hence the above discrepancy is less likely to happen in practice. As long as the (residual) CFO is considerably small, our analytical expressions yield highly accurate results. For flat-fading case, simulation results agree well with our analytical results for both $b = 0.05$ and $b = 0.1$.

We apply CFO estimation and compensation at the receiver to show the accuracy of our analytical results for practical systems. For AWGN channel, we can derive the ML CFO estimator based on the signal model in [10, eq. 1] and the ML CFO estimator based on this signal model is used in the simulation. We use an OFDM system with $N = 16$ in a quasi-static channel. In our simulation we have one OFDM preamble/training symbol followed by only one OFDM data symbol. In our analytical derivation we did not consider the CFO-induced, symbol-index-dependent phase shift of $\exp(j2\pi v\Delta m(N + N_g)/N)$ where m is the OFDM symbol index and N_g is the number of guard samples. We simply assume that every symbol is phase synchronized so that we can neglect the above phase shift. The simulation results are shown in the same Fig. 1 and we observe an excellent match between the analytical and simulation results.

The results for the frequency-flat Rayleigh fading channel are shown in Fig. 3 for both channel-independent and channel-dependent residual CFO cases. We use the CFO estimator (MLE1) from [9] in this case. The analytical results match very well with the simulation results.

VI. Conclusion

In this paper we have presented SER expressions for 4-QAM OFDM systems with random (residual) CFO. We have derived SER expressions of OFDM systems over AWGN and frequency-flat Rayleigh fading channels for both cases of channel-independent and channel-dependent random (residual) CFO. Simulations show how close our analytical results with the exact results.

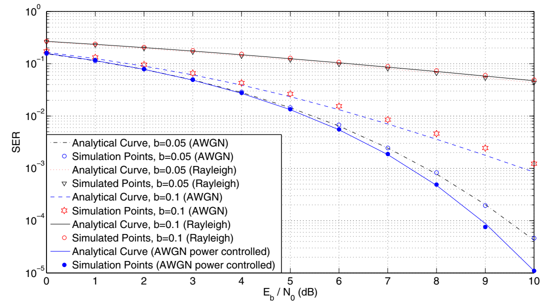

Fig. 1. SER curves for AWGN/frequency-flat Rayleigh fading channel: subcarriers (N)=8, b=0.1, b=0.05 and SER curves for AWGN channel with channel-independent (power controlled) residual CFO/CFO: N=16.

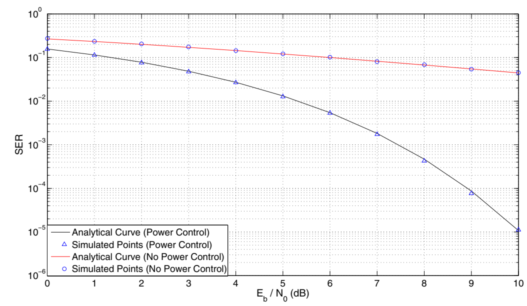

Fig. 2. SER curves for the frequency-flat Rayleigh fading channel with channel-independent (power controlled) residual CFO/CFO: N=52, and channel-dependent (no power controlled) residual CFO/CFO: N=32.

References

[1] L. Rugini and P. Banelli, "BER of OFDM systems impaired by carrier frequency offset in multipath fading channels," *IEEE Trans. Wireless. Commun.*, vol. 4, pp. 2279-2288, Sep. 2005.

[2] T. Keller and L. Hanzo, "Adaptive multicarrier modulation: A convenient framework for time-frequency processing in wireless communications," *Proc. IEEE,* vol. 88, pp. 611-640, May 2000.

[3] K. Sathananthan and C. Tellambura, "Probability of error calculation of OFDM systems with frequency offset," *IEEE Trans. Commun.*, vol. 49, pp. 1884-1888, Nov. 2001.

[4] K. D. P. Dharmawansa, R. M. A. P. Rajatheva and H. Minn, "An exact error probability analysis of OFDM systems with frequency offset," under minor revision, submitted to *IEEE Trans. Commun.*.

[5] P. C. Weeraddana and R. M. A. P. Rajatheva , "Probability of error analysis of OFDM systems with random residual frequency offset," *IEEE VTC 2007 Spring*.

[6] M. K. Simon and M. S. Alouini, *Digital Communication Over Fading Channels*, 2nd ed., A John Wiley and Sons, Inc., Publication, 2005.

[7] H. Minn, Y. Li, N. Al-Dharhir and R. Calderbank, "Pilot Designs for consistent frequency offset estimation in OFDM systems," in *Proc. IEEE ICC 2006*.

[8] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory,* Englewood Cliffs, NJ: Prentice-Hall, 1993.

[9] Michele Morelli and Umberto Mengali, "Carrier-frequency estimation for transmission over selective channels," in *IEEE Trans. Commun.*, vol. 48, pp. 1580-1589, Sept. 2000.

[10] H. Minn and S. Xing, "An optimal training signal structure for frequency offset estimation," *IEEE Trans. Commun.*, vol. 53, pp. 343-355, Feb. 2005.

[11] S. Verdu, *Multiuser Detection,* Cambridge University Press, 1998.

$$I_2^*(t_0, t_1, t_2) = \frac{1}{2} - \frac{\psi_1+\psi_2}{2\pi} - \frac{\varepsilon_1}{2\pi\sqrt{1+\varepsilon_1^2}}\left(\frac{\pi}{2} - \tan^{-1}\left[\frac{t_0\sigma_R a_1}{\sqrt{1+\varepsilon_1^2}}\right]\right) - \frac{\varepsilon_2}{2\pi\sqrt{1+\varepsilon_2^2}}\left(\frac{\pi}{2} - \tan^{-1}\left[\frac{t_0\sigma_R a_2}{\sqrt{1+\varepsilon_2^2}}\right]\right). \tag{30}$$