An Exact Error Probability Analysis of OFDM Systems with Frequency Offset

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***Abstract*—In this paper, we derive exact closed form bit error rate (BER) or symbol error rate (SER) expressions for orthogonal frequency division multiplexing (OFDM) systems with carrier frequency offset (CFO). We consider the performance of an OFDM system subject to CFO error in additive white Gaussian noise (AWGN), frequency flat and frequency selective Rayleigh fading channels. The BER/ SER performances of BPSK and QPSK modulation schemes are analyzed for AWGN and frequency-flat Rayleigh fading channels while BPSK is considered for frequency-selective Rayleigh fading channels. Our results can easily be reduced to the respective analytical error rate expressions for the OFDM systems without CFO error. Furthermore, the simulation results are provided to verify the accuracy of the new error rate expressions.**

***Index Terms*—Bit error rate (BER), frequency offset, frequency selective fading, orthogonal frequency division multiplexing (OFDM), Rayleigh fading, symbol error rate (SER).**

I. Introduction

ORTHOGONAL frequency division multiplexing (OFDM) is a promising technology for broadband wireless communications systems due to its low complexity equalization capability in frequency selective fading channels and its adaptability/scalability to the channel conditions. By definition, OFDM expects the subcarriers to be orthogonal. But, the factors such as carrier frequency mismatching, time variations due to Doppler shift or phase noise usually eliminate the orthogonality of the subcarriers. This gives rise to intercarrier interference (ICI) which degrades the performance of OFDM systems significantly [1].

A careful literature survey reveals two most widely used methods of error performance analysis of OFDM systems. One method is to approximate the ICI as a Gaussian process based on the central limit theorem [2]- [4]. However, in [5], Keller and Hanzo showed that such approximation leads to wrong results (in their own words "pessimistic") for high signal to noise ratios. On the other hand, Pollet *et al* [6] derive analytically the signal to noise ratio degradation due to the ICI. Zhao and Häggman [7] use the moments of ICI distribution to come up with a more accurate BER result. The approach due to Sathananthan and Tellambura [8] is unique in comparison with all the above approaches since they use characteristic functions and Beaulieu series [9] to derive exact BER expressions in the presence of ICI. However, their exact results are given only for the AWGN channel in the form of infinite series expansions. Recently Beaulieu and Tan [10] devise a characteristic function based approach to analyze the exact BER of OFDM systems with receiver windowing.

We follow the procedure presented in Sathananthan and Tellambura [8] and Rugini and Banelli [3] with different mathematical insight to come up with exact closed form BER/SER expressions for OFDM with ICI over AWGN, frequency selective and flat fading channels. The important thing to be noted here is that we do not follow the Gaussian approximation of the ICI, instead we show that the probability density function of ICI is a mixture of Gaussian densities with properly selected parameters. The standard error rate formulae without ICI are shown to be some special cases of our more general results.

The paper is organized as follows. Section II derives the exact closed form BER expressions for frequency flat fading channel and AWGN channel. The novel derivation of BER expression for frequency selective Rayleigh fading channels is given in Section III. Selected numerical results are provided in Section IV to verify the accuracy of our derivation. Section V concludes the paper with some final remarks.

II. Exact Error Rate Expressions for AWGN and Flat Fading Channels with CFO

We assume perfect quasi-static flat fading channel. The sampled received signal on the kth subcarrier after going through the fast Fourier transform processor is given by [3]

$$r_k = \alpha X_k S_1 + \alpha \sum_{l=1, l\neq k}^{N} S_{l-k+1} X_l + n_k, \quad k =, 1, 2, .., N \quad (1)$$

where X_k is the transmitted symbol on the kth subcarrier, α denotes the complex fading coefficient which can be modelled as a circularly-symmetric complex Gaussian variable having zero mean and variance σ_R^2 per dimension, n_k is the additive white Gaussian noise modelled as a zero mean complex Gaussian random variable having variance σ^2 per dimension and N is the number of subcarriers. The ICI coefficients, S_k

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is given by [3]

$$S_k = \frac{\sin(\pi[k-1+\epsilon])}{N\sin\left(\frac{\pi[k-1+\epsilon]}{N}\right)}\exp\left\{j\pi\left(1-\frac{1}{N}\right)(k-1+\epsilon)\right\} \tag{2}$$

where ϵ is the normalized (by the subcarrier spacing) frequency offset. One should note that, (1) becomes an ideal AWGN channel if we set $\alpha = 1$. Without loss of generality, we assume equiprobable message symbols and consider error rates for the first subcarrier. Before starting the main derivation, we present an important identity between product and sum of cosines as

$$\prod_{k=1}^{M}\cos(\phi_k) \equiv \frac{1}{2^{M-1}}\sum_{k=1}^{2^{M-1}}\cos\left(\mathbf{\Phi}^T\mathbf{e}_k\right) \tag{3}$$
$$\equiv \frac{1}{2^M}\sum_{k=1}^{2^{M-1}}\exp\left(j\mathbf{\Phi}^T\mathbf{e}_k\right)+\exp\left(-j\mathbf{\Phi}^T\mathbf{e}_k\right)$$

where $\mathbf{\Phi} = (\phi_1\ \phi_2\ \ldots\ \phi_M)^T$, $\mathbf{e}_k$ is the kth column of a more general $M\times 2^{M-1}$ matrix $\mathbf{E}_M$, $j=\sqrt{-1}$ and $(\cdot)^T$ denotes the transpose of a matrix. The kth column of $\mathbf{E}_M$ is essentially the binary representation of the number 2^M-k, where zeros are replaced with -1s. The matrix $\mathbf{E}_M$ for the value of $M=4$ can be written as

$$\mathbf{E}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1\\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}.$$

The identity in (3) can be verified very easily by repeatedly applying the identity $\cos C\cos D \equiv \frac{1}{2}[\cos(C+D)+\cos(C-D)]$ to the product of two cosine terms taken at a time, in the left side of (3).

A. BPSK Modulation

1) AWGN Channel: For BPSK modulation $X_k \in \{1,-1\}$ and we consider the first subcarrier with the transmitted symbol 1. Since the constellation is purely real, only the real part of (1) will be considered. Following [8], we obtain the characteristic function (CHF) of the real part of r_1, $\Re(r_1)$, as

$$\phi_{\Re(r_1)}(\omega) = \exp\left\{j\omega\Re(S_1)-\frac{\omega^2\sigma^2}{2}\right\}\prod_{l=2}^{N}\cos(\omega\Re[S_l]) \tag{4}$$

which can be further simplified by using (3) as

$$\phi_{\Re(r_1)}(\omega) = \frac{1}{2^{N-1}}\sum_{k=1}^{2^{N-2}}\left(\exp\left\{j\omega\theta_k-\frac{\omega^2\sigma^2}{2}\right\}\right. \tag{5}$$
$$\left.+\exp\left\{j\omega\beta_k-\frac{\omega^2\sigma^2}{2}\right\}\right)$$

where $\mathbf{S} = (S_2\ S_3\ \ldots\ S_N)^T$, $\mathbf{E}_{N-1}$ is of dimension $(N-1)\times 2^{N-2}$, $\theta_k = \Re\left(S_1+\mathbf{S}^T\mathbf{e}_k\right)$, and $\beta_k = \Re\left(S_1-\mathbf{S}^T\mathbf{e}_k\right)$. It is obvious that (5) represents the CHF of a mixture of Gaussian density functions. Since the hypotheses are binary, an error occurs if $\Re(r_1) < 0$. Now, the bit error probability can be written as

$$P_b(\xi) = \frac{1}{2^{N-1}}\sum_{k=1}^{2^{N-2}}\left\{Q\left(\sqrt{2\gamma}\theta_k\right)+Q\left(\sqrt{2\gamma}\beta_k\right)\right\} \tag{6}$$

where $\gamma = \frac{E_b}{N_0}$, $Q(x)$ is the Gaussian Q function. One should note that in our case of interest $E_b = 1$ and $\sigma^2 = \frac{N_0}{2}$ with N_0 being the noise power spectral density. If we let $\epsilon = 0$, then $\theta_k = \beta_k = 1$ and it follows that (6) simplifies to [11,eq. 8.3].

2) Rayleigh Flat Fading Channel: In the Rayleigh flat fading channel with coherent detection, the conditional bit error probability can be written using (6) as

$$P_b(\xi|\alpha) = \frac{1}{2^{N-1}}\sum_{k=1}^{2^{N-2}}\left\{Q\left(\sqrt{2\gamma}\theta_k|\alpha|\right)+Q\left(\sqrt{2\gamma}\beta_k|\alpha|\right)\right\} \tag{7}$$

where $|z|$ denotes the absolute value of z. The bit error probability is calculated as [11,eq. 8.102]

$$P_b(\xi) = \int_0^\infty P_b(\xi|\alpha)\, f(|\alpha|)\, d|\alpha| \tag{8}$$

where $|\alpha|$ has a Rayleigh distribution given by $f(|\alpha|) = \frac{|\alpha|}{\sigma_R^2}\exp\left(-\frac{|\alpha|^2}{2\sigma_R^2}\right)$. After applying the following Craig's formula [12]

$$Q(x) = \frac{1}{\pi}\int_0^{\pi/2}\exp\left(-\frac{x^2}{2\sin^2\psi}\right)d\psi, \qquad x>0, \tag{9}$$

to (7) and after some algebraic manipulations, (8) gives the bit error probability as

$$P_b(\xi) = \frac{1}{2}-\frac{1}{2^N}\sum_{k=1}^{2^{N-2}}\left\{\sqrt{\frac{2\sigma_R^2\theta_k^2\gamma}{1+2\sigma_R^2\theta_k^2\gamma}}+\sqrt{\frac{2\sigma_R^2\beta_k^2\gamma}{1+2\sigma_R^2\beta_k^2\gamma}}\right\}. \tag{10}$$

This leads to [11,eq. 8.104] when $\epsilon = 0$.

B. QPSK Modulation

1) AWGN Channel: As usual we consider the first subcarrier with the transmitted symbol $X_1 = 1+j$ with $X_k = \{\pm 1 \pm j\}$ and we define

$$u = X_1S_1+\sum_{l=2}^{N}S_lX_l. \tag{11}$$

The probability of a correct decision is the probability that $u+n_1$ lies inside D_1, the first quadrant of the complex plane, and it is given in [8] as

$$P(u+n_1\in D_1|X_1 = 1+j, u) = Q\left(\frac{-\Re(u)}{\sigma}\right)Q\left(\frac{-\Im(u)}{\sigma}\right) \tag{12}$$

where $\Im(z)$ is the imaginary part of z. The average correct symbol decision probability is obtained by averaging (12) over u for which the required pdf of u is derived below. Since the variable u is two dimensional and $X_1 = 1+j$, its two dimensional CHF can be written as [8]

$$\phi(\omega_I,\omega_Q) = e^{(j\omega_I[\Re(S_1)-\Im(S_1)]+j\omega_Q[\Im(S_1)+\Re(S_1)])} \tag{13}$$
$$\times\prod_{l=2}^{N}\Big\{\cos(\omega_I\Re(S_l)+\omega_Q\Im(S_l))$$
$$\times\cos(\omega_I\Im(S_l)-\omega_Q\Re(S_l))\Big\}.$$

More compact version of (13) can be written using the vector notations as

$$\phi\left(\omega_I,\omega_Q\right)=e^{\left\{j\mathbf{\Omega}^T\left(\mathbf{S}_A^1-\mathbf{S}_B^1\right)\right\}}\prod_{l=2}^{N}\cos\left(\mathbf{\Omega}^T\mathbf{S}_A^l\right)\cos\left(\mathbf{\Omega}^T\mathbf{S}_B^l\right) \tag{14}$$

where $\mathbf{\Omega}=\left(\omega_I\ \omega_Q\right)^T, \mathbf{S}_A^l=\left(\Re\left(S_l\right)\ \Im\left(S_l\right)\right)^T$ and $\mathbf{S}_B^l=\left(\Im\left(S_l\right)\ -\Re\left(S_l\right)\right)^T$ for all $l=1,2,...,N$. Then we use (3) to express (14) as

$$\phi\left(\omega_I,\omega_Q\right)=\frac{1}{2^{2(N-2)}}\exp\left\{j\mathbf{\Omega}^T\left(\mathbf{S}_A^1-\mathbf{S}_B^1\right)\right\} \times\sum_{k=1}^{2^{N-2}}\sum_{n=1}^{2^{N-2}}\cos\left(\mathbf{\Omega}^T\mathbf{S}_A\mathbf{e}_k\right)\cos\left(\mathbf{\Omega}^T\mathbf{S}_B\mathbf{e}_n\right) \tag{15}$$

where $\mathbf{S}_A=\left(\mathbf{S}_A^2\ \mathbf{S}_A^3\ \ldots\ \mathbf{S}_A^N\right), \mathbf{S}_B=\left(\mathbf{S}_B^2\ \mathbf{S}_B^3\ \ldots\ \mathbf{S}_B^N\right), \mathbf{e}_k$ and $\mathbf{e}_n$ are column vectors taken from more general matrix $\mathbf{E}_{N-1}$. By using Euler's relationship with some rearrangements of terms, (15) can be written as

$$\phi\left(\omega_I,\omega_Q\right)=\frac{1}{2^{2(N-1)}}\sum_{k=1}^{2^{N-2}}\sum_{n=1}^{2^{N-2}}\left\{e^{\left(j\mathbf{\Omega}^T\mathbf{C}_1\right)}+e^{\left(j\mathbf{\Omega}^T\mathbf{C}_2\right)}+e^{\left(j\mathbf{\Omega}^T\mathbf{C}_3\right)}+e^{\left(j\mathbf{\Omega}^T\mathbf{C}_4\right)}\right\} \tag{16}$$

where $\mathbf{C}_1=\mathbf{S}_A^1-\mathbf{S}_B^1+\mathbf{S}_A\mathbf{e}_k+\mathbf{S}_B\mathbf{e}_l, \mathbf{C}_2=\mathbf{S}_A^1-\mathbf{S}_B^1-\mathbf{S}_A\mathbf{e}_k-\mathbf{S}_B\mathbf{e}_l, \mathbf{C}_3=\mathbf{S}_A^1-\mathbf{S}_B^1+\mathbf{S}_A\mathbf{e}_k-\mathbf{S}_B\mathbf{e}_l$ and $\mathbf{C}_4=\mathbf{S}_A^1-\mathbf{S}_B^1-\mathbf{S}_A\mathbf{e}_k+\mathbf{S}_B\mathbf{e}_l$. Thus, the Fourier transform of (16) yields the two dimensional pdf of u as

$$p\left(\Re(u),\Im(u)\right)=\frac{1}{2^{2(N-1)}}\sum_{k=1}^{2^{N-2}}\sum_{n=1}^{2^{N-2}}\sum_{m=1}^{4}\delta\left\{\Re(u)-\psi_{kn}[1,m]\right\}\times\delta\left\{\Im(u)-\psi_{kn}[2,m]\right\} \tag{17}$$

where $\psi_{kl}[p,q]$ is the (p,q)th element of the 2×4 matrix $\mathbf{\Psi}$ defined as $\mathbf{\Psi}=\left(\mathbf{C}_1\ \mathbf{C}_2\ \mathbf{C}_3\ \mathbf{C}_4\right)$ and $\delta(x)$ is the Dirac delta function. Averaging (12) with respect to the pdf given in (17) yields the SER as

$$P_s\left(\xi\right)=1-\frac{1}{2^{2N-2}}\sum_{k=1}^{2^{N-2}}\sum_{n=1}^{2^{N-2}}\sum_{m=1}^{4}Q\left(-\sqrt{2\gamma}\psi_{kn}[1,m]\right)\times Q\left(-\sqrt{2\gamma}\psi_{kn}[2,m]\right). \tag{18}$$

In the absence of ICI (i.e. $\epsilon=0$), $\mathbf{\Psi}$ can be expressed as $\mathbf{\Psi}=\begin{pmatrix}1&1&1&1\\1&1&1&1\end{pmatrix}$ since $S_k=0$ for all $k\neq 1$ and $S_1=1$. Upon substituting the elements of the new $\mathbf{\Psi}$ matrix into (18), our result in (18) reduces to [11,eq. 8.20].

2) Rayleigh Flat Fading Channel: The conditional symbol error rate in the case of coherent detection can be expressed as

$$P_s\left(\xi|\alpha\right)=1-\frac{1}{2^{2N-2}}\sum_{k=1}^{2^{N-2}}\sum_{n=1}^{2^{N-2}}\sum_{m=1}^{4}Q\left(-\sqrt{2\gamma}\psi_{kn}[1,m]|\alpha|\right)\times Q\left(-\sqrt{2\gamma}\psi_{kn}[2,m]|\alpha|\right) \tag{19}$$

where $|\alpha|$ is Rayleigh distributed. We encounter integrals of the form

$$\int_0^\infty Q\left(\lambda|\alpha|\right)Q\left(\mu|\alpha|\right)f\left(|\alpha|\right)d|\alpha|, \int_0^\infty Q\left(\lambda|\alpha|\right)f\left(|\alpha|\right)d|\alpha|$$

in averaging (19) and they can be solved using [11,eq. 4.8], [11,eq. 5.102] and [11,eq. 5.6] to give the SER in (20) as given at the bottom. Equation (20) can easily be simplified to [11,eq. 8.108] if we let $\epsilon=0$.

III. Exact Error Rate Expression in Frequency Selective Fading Channels

Here we assume L sample-spaced tap-delay-line model for the frequency selective channel and the time domain tap coefficients $h_l,\ l=1,2,...,L$ are modeled as zero mean complex Gaussian random variables having variances $\{\sigma_{h_l}^2\}$ with $\sigma_{h_1}^2+\sigma_{h_2}^2+\cdots+\sigma_{h_L}^2=1$. Furthermore the channel is assumed to be quasi-static. The received kth subcarrier symbol is given by [3]

$$r_k=\alpha_k X_k S_1+\sum_{m=1,m\neq k}^{N}\alpha_m S_{m-k+1}X_m+n_k, k=1,2,..,N \tag{21}$$

where $\left(\alpha_1\ \alpha_2\ \cdots\ \alpha_N\right)^T=\mathbf{F}_L\mathbf{h}$ with $\mathbf{h}=\left(h_1\ h_2\ \cdots\ h_L\right)^T$, and $\mathbf{F}_L$ denotes the first L columns of the $N\times N$ discrete Fourier transform (DFT) matrix, defined by $F_L[p,q]=\exp\left\{-j2\pi\frac{(p-1)(q-1)}{N}\right\}$. For BPSK modulation $X_k=\{+1,-1\}$, and as usual, we consider the first subcarrier with the transmitted symbol $X_1=1$. Then (21) can be written as

$$r_1=\alpha_1 S_1+\sum_{m=2}^{N}\alpha_m S_m X_m+n_1 \tag{22}$$

and the decision variable can be formed as $\Re(\overline{\alpha}_1 r_1)$, where $\overline{z}$ denotes the complex conjugate of z. Now we can rewrite the decision variable as

$$\Re(\overline{\alpha}_1 r_1)=\Re\left(S_1\right)|\alpha_1|^2+\sum_{m=2}^{N}\Re\left(\overline{\alpha}_1\alpha_m S_m\right)X_m+\Re\left(\overline{\alpha}_1 n_1\right). \tag{23}$$

$$P_s\left(\xi\right)=\frac{3}{4}-\frac{1}{2^{2N-1}}\sum_{k=1}^{2^{N-2}}\sum_{n=1}^{2^{N-2}}\sum_{m=1}^{4}\sqrt{\frac{2\sigma_R^2\psi_{kn}^2[1,m]\gamma}{1+2\sigma_R^2\psi_{kn}^2[1,m]\gamma}}\left(1-\frac{1}{\pi}\arctan\left\{\frac{\psi_{kn}[1,m]}{\psi_{kn}[2,m]}\sqrt{1+\frac{1}{2\sigma_R^2\psi_{kn}^2[1,m]\gamma}}\right\}\right)+\sqrt{\frac{2\sigma_R^2\psi_{kn}^2[2,m]\gamma}{1+2\sigma_R^2\psi_{kn}^2[2,m]\gamma}}\left(1-\frac{1}{\pi}\arctan\left\{\frac{\psi_{kn}[2,m]}{\psi_{kn}[1,m]}\sqrt{1+\frac{1}{2\sigma_R^2\psi_{kn}^2[2,m]\gamma}}\right\}\right) \tag{20}$$

Following the same line of arguments as before, we can write the conditional CHF of the random variable $\Re\left(\overline{\alpha}_1 r_1\right)|\alpha_1, \boldsymbol{\alpha}$ as

$$\phi_{\Re(\overline{\alpha}_1 r_1)|\alpha_1,\boldsymbol{\alpha}}(\omega) = \frac{1}{2^{N-1}} \times \sum_{k=1}^{2^{N-2}} \left(\exp\left\{ j\omega\left[|\alpha_1|^2\Re\left(S_1\right) + \Re\left(\overline{\alpha}_1 \mathbf{P}_k^T \boldsymbol{\alpha}\right)\right] - \frac{\sigma_n^2\omega^2}{2}\right\} + \exp\left\{ j\omega\left[|\alpha_1|^2\Re\left(S_1\right) - \Re\left(\overline{\alpha}_1 \mathbf{P}_k^T \boldsymbol{\alpha}\right)\right] - \frac{\sigma_n^2\omega^2}{2}\right\}\right) \tag{24}$$

where $\boldsymbol{\alpha} = \left(\alpha_2\ \alpha_3\ \cdots\ \alpha_N\right)^T$, $\mathbf{P}_k = \operatorname{diag}\left(S_2, S_3, \cdots, S_N\right)\mathbf{e}_k$ with $\operatorname{diag}\left(\cdot\right)$ being a diagonal matrix and $\sigma_n^2 = |\alpha_1|^2\sigma^2$. It is obvious that (24) represents the CHF of a mixture of Gaussian densities. Now, an error occurs if $\Re\left(\overline{\alpha}_1 r_1\right)|\alpha_1, \boldsymbol{\alpha} < 0$. Thus the conditional BER can be written as

$$P_b\left(\xi|\alpha_1, \boldsymbol{\alpha}\right) = \frac{1}{2^{N-1}}\sum_{k=1}^{2^{N-2}} Q\left(\frac{|\alpha_1|^2\Re\left(S_1\right) + \Re\left(\overline{\alpha}_1 \mathbf{P}_k^T \boldsymbol{\alpha}\right)}{\sigma_n}\right) + Q\left(\frac{|\alpha_1|^2\Re\left(S_1\right) - \Re\left(\overline{\alpha}_1 \mathbf{P}_k^T \boldsymbol{\alpha}\right)}{\sigma_n}\right). \tag{25}$$

Following [3] we calculate the unconditional BER as

$$P_b\left(\xi\right) = \int_{\alpha_1}\int_{\boldsymbol{\alpha}} P_b\left(\xi|\alpha_1, \boldsymbol{\alpha}\right) f_{\boldsymbol{\alpha}|\alpha_1}\left(\boldsymbol{\alpha}|\alpha_1\right) d\boldsymbol{\alpha} f_{\alpha_1}\left(\alpha_1\right) d\alpha_1 \tag{26}$$

where the conditional density $f_{\boldsymbol{\alpha}|\alpha_1}\left(\boldsymbol{\alpha}|\alpha_1\right)$ is Gaussian with mean $E\left(\boldsymbol{\alpha}|\alpha_1\right)$ and covariance $\mathbf{C}_{\boldsymbol{\alpha}|\alpha_1}$ defined by [3] [12]

$$\begin{aligned} E\left(\boldsymbol{\alpha}|\alpha_1\right) &= \alpha_1 c_{\alpha_1\alpha_1}^{-1}\mathbf{C}_{\boldsymbol{\alpha}\alpha_1} \\ \mathbf{C}_{\boldsymbol{\alpha}|\alpha_1} &= \mathbf{C}_{\boldsymbol{\alpha}\boldsymbol{\alpha}} - c_{\alpha_1\alpha_1}^{-1}\mathbf{C}_{\boldsymbol{\alpha}\alpha_1}\mathbf{C}_{\boldsymbol{\alpha}\alpha_1}^H \end{aligned} \tag{27}$$

with $E(\cdot)$ and $(\cdot)^H$ denoting the statistical expectation and the Hermitian transpose operation, respectively, $c_{\alpha_l\alpha_m} = E\left(\alpha_l\overline{\alpha}_m\right), 1 \leq l, m \leq N$, and $\mathbf{C}_{\boldsymbol{\alpha}\alpha_1} = \left(c_{\alpha_2\alpha_1}\ c_{\alpha_3\alpha_1}\ \cdots\ c_{\alpha_N\alpha_1}\right)^T$. The matrix $\mathbf{C}_{\boldsymbol{\alpha}\boldsymbol{\alpha}}$ is obtained from the frequency domain channel covariance matrix defined as

$$\begin{aligned} \mathbf{C} &= E\left\{\left(\alpha_1\ \boldsymbol{\alpha}^T\right)^T\left(\overline{\alpha}_1\ \boldsymbol{\alpha}^H\right)\right\} = \begin{pmatrix} c_{\alpha_1\alpha_1} & \mathbf{C}_{\boldsymbol{\alpha}\alpha_1}^H \\ \mathbf{C}_{\boldsymbol{\alpha}\alpha_1} & \mathbf{C}_{\boldsymbol{\alpha}\boldsymbol{\alpha}} \end{pmatrix} \\ &= \mathbf{F}_L\mathbf{C}_h\mathbf{F}_L^H \end{aligned}$$

where $\mathbf{C}_h$ is the time domain channel covariance matrix. Evaluation of the multidimensional integral in (26) is an arduous task. Hence we propose the following alternative formulation which avoids much of the complexity associated with (26).

Let us consider the random variable $z_k = \Re\left(\overline{\alpha}_1\mathbf{P}_k^T\boldsymbol{\alpha}\right)$. The conditional random variable $z_k|\alpha_1$ is Gaussian with mean and variance to be determined. Using (27) we obtain

$$E\left(z_k|\alpha_1\right) = |\alpha_1|^2 c_{\alpha_1\alpha_1}^{-1}\Re\left(\mathbf{P}_k^T\mathbf{C}_{\boldsymbol{\alpha}\alpha_1}\right) = |\alpha_1|^2 a_k \tag{28}$$

where $a_k = c_{\alpha_1\alpha_1}^{-1}\Re\left(\mathbf{P}_k^T\mathbf{C}_{\boldsymbol{\alpha}\alpha_1}\right)$. Following [12] and [13] with some algebraic manipulations, we obtain the variance of $z_k|\alpha_1$ as

$$\operatorname{Var}\left(z_k|\alpha_1\right) = \frac{1}{2}|\alpha_1|^2\mathbf{P}_k^T\mathbf{C}_{\boldsymbol{\alpha}|\alpha_1}\overline{\mathbf{P}}_k = \frac{1}{2}|\alpha|^2 b_k \tag{29}$$

where $\overline{\mathbf{P}}_k$ denotes the conjugate of $\mathbf{P}_k$ and $b_k = \mathbf{P}_k^T\mathbf{C}_{\boldsymbol{\alpha}|\alpha_1}\overline{\mathbf{P}}_k$. One should note that a_k and b_k are introduced for the notational simplicity. Hence we can conclude that $z_k|\alpha_1$ is a Gaussian random variable with mean and variance given by $|\alpha_1|^2 a_k$ and $\frac{1}{2}|\alpha_1|^2 b_k$, respectively. Now we can rewrite (25) as

$$P_b\left(\xi|\alpha_1, \boldsymbol{\alpha}\right) = \frac{1}{2^{N-1}}\sum_{k=1}^{2^{N-2}}\left\{Q\left(\frac{|\alpha_1|^2\Re\left(S_1\right) + z_k}{\sigma_n}\right) + Q\left(\frac{|\alpha_1|^2\Re\left(S_1\right) - z_k}{\sigma_n}\right)\right\} \tag{30}$$

and rearrange the arguments of the Q functions to yield

$$P_b\left(\xi|\alpha_1, \boldsymbol{\alpha}\right) = \frac{1}{2^{N-1}}\sum_{k=1}^{2^{N-2}}\left\{Q\left(\mu_{+k} + \lambda_k Y_k\right) + Q\left(\mu_{-k} - \lambda_k Y_k\right)\right\} \tag{31}$$

where $\mu_{+k} = \frac{|\alpha_1|\left(\Re(S_1) + a_k\right)}{\sigma}, \mu_{-k} = \frac{|\alpha_1|\left(\Re(S_1) - a_k\right)}{\sigma}, \lambda_k = \sqrt{\frac{b_k}{2\sigma^2}}$ and $Y_k = \sqrt{\frac{2}{b_k}}\frac{\left(z_k - |\alpha_1|^2 a_k\right)}{|\alpha_1|}$. Furthermore, $Y_k|\alpha_1$ is a Gaussian random variable with zero mean and unit variance. Averaging (31) with respect to the conditional Gaussian random variables $Y_k|\alpha_1$ and using [15, eq.3.66], we get

$$P_b\left(\xi|\alpha_1\right) = \frac{1}{2^{N-1}}\sum_{k=1}^{2^{N-2}} Q\left(\frac{|\alpha_1|\left[\Re\left(S_1\right) + a_k\right]}{\sigma\sqrt{1 + \frac{b_k}{2\sigma^2}}}\right) + Q\left(\frac{|\alpha_1|\left[\Re\left(S_1\right) - a_k\right]}{\sigma\sqrt{1 + \frac{b_k}{2\sigma^2}}}\right). \tag{32}$$

Averaging (32) over the Rayleigh random variable $|\alpha_1|$ with pdf $f_{|\alpha_1|}\left(|\alpha_1|\right) = \frac{2|\alpha_1|}{c_{\alpha_1\alpha_1}}\exp\left(-\frac{|\alpha_1|^2}{c_{\alpha_1\alpha_1}}\right)$, we obtain the BER as

$$P_b\left(\xi\right) = \frac{1}{2} - \frac{1}{2^N}\sum_{k=1}^{2^{N-2}}\sqrt{\frac{c_{\alpha_1\alpha_1}\gamma\left(\Re\left[S_1\right) + a_k\right]^2}{1 + \gamma\left(c_{\alpha_1\alpha_1}\left[\Re\left(S_1\right) + a_k\right]^2 + b_k\right)}} + \sqrt{\frac{c_{\alpha_1\alpha_1}\gamma\left[\Re\left(S_1\right) - a_k\right]^2}{1 + \gamma\left(c_{\alpha_1\alpha_1}\left[\Re\left(S_1\right) - a_k\right]^2 + b_k\right)}}. \tag{33}$$

Further simplification of (33) can be obtained if all time domain tap coefficients are independent of each other as

$$P_b\left(\xi\right) = \frac{1}{2} - \frac{1}{2^N}\sum_{k=1}^{2^{N-2}}\sqrt{\frac{\gamma\left[\Re\left(S_1\right) + a_k\right]^2}{1 + \gamma\left(\left[\Re\left(S_1\right) + a_k\right]^2 + b_k\right)}} + \sqrt{\frac{\gamma\left[\Re\left(S_1\right) - a_k\right]^2}{1 + \gamma\left(\left[\Re\left(S_1\right) - a_k\right]^2 + b_k\right)}} \tag{34}$$

where the modified parameters a_k and b_k are given by $a_k = \Re\left(\mathbf{P}_k^T\mathbf{C}_{\boldsymbol{\alpha}\alpha_1}\right)$ and $b_k = \mathbf{P}_k^T\left(\mathbf{C}_{\boldsymbol{\alpha}\boldsymbol{\alpha}} - \mathbf{C}_{\boldsymbol{\alpha}\alpha_1}\mathbf{C}_{\boldsymbol{\alpha}\alpha_1}^H\right)\overline{\mathbf{P}}_k$. For a frequency flat Rayleigh fading channel, we have $\boldsymbol{\alpha} = \alpha_1\left(1\,1\ \cdots\ 1\right)^T$ with $a_k = \Re\left(\mathbf{S}^T\mathbf{e}_k\right)$ and $b_k = 0$. By substituting these values in (33) and observing the fact that $c_{\alpha_1\alpha_1} = 2\sigma_R^2$, we can see that (33) reduces to (10).

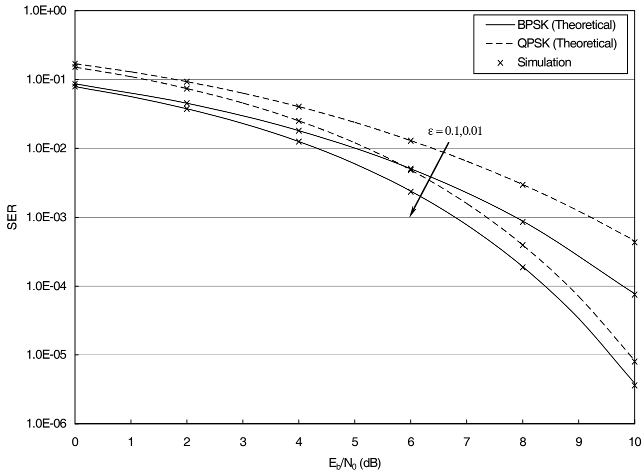

Fig. 1. Probability of symbol error for BPSK/QPSK over AWGN channel with $N = 8$.

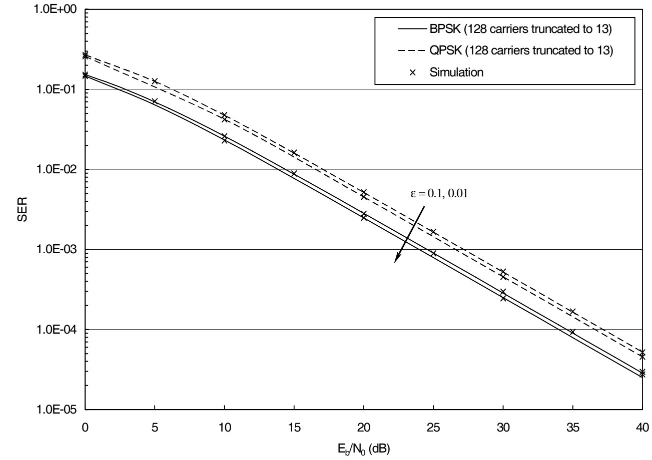

Fig. 3. Approximate performance of BPSK/QPSK over Rayleigh flat fading channel with $N = 128$ and $\sigma_R^2 = 0.5$.

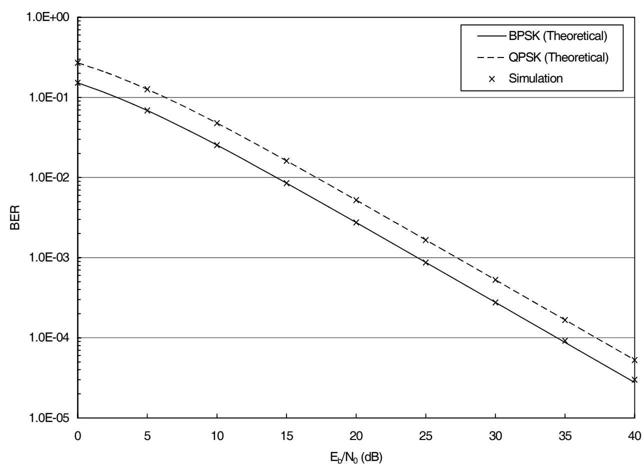

Fig. 2. Probability of symbol error for BPSK/QPSK over Rayleigh flat fading channel with $N = 8, \epsilon = 0.1$ and $\sigma_R^2 = 0.5$.

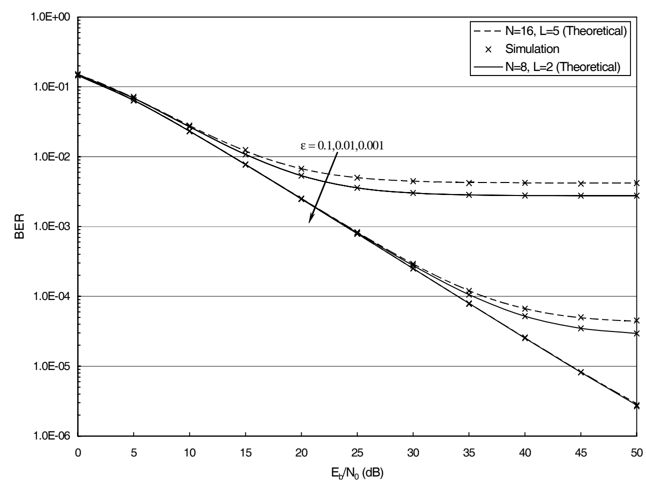

Fig. 4. BER for BPSK over frequency selective Rayleigh fading channel.

IV. Simulation Results and Discussion

Fig. 1 presents SER of an OFDM system with $N = 8$ for BPSK/QPSK modulation schemes over AWGN channel. The simulation results agree with our analytical results in (6) and (18). The SER performance of an OFDM system with BPSK/QPSK modulation and $N = 8$ over Rayleigh flat fading channel is shown in Fig. 2. The accuracy of the newly derived SER expressions in (10) and (20) for QPSK modulation is thus verified by the simulation results.

Careful inspection of the SER/BER formulae reveals that the computational complexity increases exponentially with the number of subcarriers. However, (2) shows that only a couple of subcarriers introduce the significant interference terms (It should be noted that S_k is a periodic sequence with a period of N). Based on these observations, a compromise can be reached between the computational complexity and mathematical exactness by truncating the number of ICI terms involved in the formulae in calculating the exact results. Fig. 3 shows the SER performance of an OFDM system with $N = 128$ subcarriers and BPSK/QPSK modulation schemes over frequency flat Rayleigh fading channel, where the analytical results are obtained by using 13 ICI terms (i.e., $S_{-6}, S_{-5}, \ldots, S_6$). As can be seen, the truncation does not introduce much deviation from the real performance given by simulation results. Note that one can truncate at a larger number of ICI terms to obtain a more accurate result at the expense of computational rigour.

Fig. 4 depicts the BER performance of an OFDM system with $N = 8, 16$ subcarriers and BPSK modulation for $\epsilon = 0.1, 0.01, 0.001$ in a frequency selective Rayleigh fading channel having $L = 2, 5$ taps. For brevity we assume uniform power delay profile characteristics for the frequency selective channel. The theoretical BER values coincide exactly with their simulation counterparts verifying the exactness of our new BER formulation.

The BER formula for the frequency selective channel also has an exponential complexity, which imposes a practical bound of using the formula for a large number of subcarriers, e.g. 256. Hence it is natural to limit the number of ICI terms

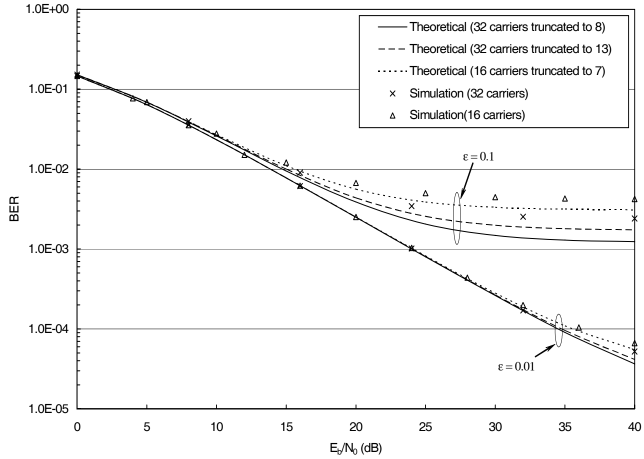

Fig. 5. BER performance with ICI truncation over frequency selective Rayleigh fading channel having $L = 5$.

and reduce the complexity of calculations at the expense of acceptable accuracy loss in the exact BER result. Fig. 5 shows the analytical BER results obtained with ICI truncation for the frequency selective Rayleigh fading channel. At low frequency offset and E_b/N_0 value up to 30 dB (sufficient for all practical systems), there is no significant BER difference. For relatively large frequency offset, there is a considerable BER discrepancy at high SNR, suggesting about half of ICI terms to be included in the calculation.

The mathematical complexity of the derived BER/SER expressions imposes limitations on the practical applicability of them. However, it is obvious that the analytical expressions are easy to evaluate when the number of subcarriers are sufficiently low (i.e. $N = 8, 16, 32$). For fairly high number of subcarriers, the exponential computational complexity associated with the analytical results suggests the use of simulation based approach to find the approximate BER/SER values. This is a tradeoff between computational efficiency and analytical exactness of our approach. Furthermore, we have witnessed that the method of truncation serves as an alternative to time consuming simulations. The strict limitation of number of ICI terms by truncation introduces a perceived performance gain which is extremely low in low SNR region. Alternatively, for small values of ϵ we can select an arbitrarily small number of ICI terms without introducing a performance gain even in high SNR region. As a result we can say that the analytical results are useful in its exact form or in the truncated form if either the number of subcarriers or ϵ is low.

V. Conclusion

In this paper, we derive exact closed form BER/SER expressions for OFDM systems with ICI over AWGN, frequency flat and selective Rayleigh fading channels. The performance of BPSK and QPSK modulation schemes are considered with the new mathematical formulation introduced here for frequency flat and AWGN channels, while only the BPSK performance is considered for frequency selective channel. Our results are of importance since no exact analytical results are reported in the open literature for the performance of OFDM systems with frequency offset over either flat or selective channels. The new formulae can easily be simplified to give the corresponding results available in the literature for ICI free cases. The computational complexity of the exact results increases exponentially with the increase of the number of subcarriers. A lower complexity version can be obtained by truncating the ICI coefficients. The discrepancy due to this truncation is considerable for frequency selective fading channels in high SNR region with large frequency offset, but insignificant for frequency flat fading channels and frequency selective fading channels with low frequency offset. As a rule of thumb we can use a half of the total ICI terms to obtain an acceptable accuracy in BER when a large number of carriers are employed.

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