An Exact Error Probability Analysis of OFDM Systems with Frequency Offest

Prathapasinghe Dharmawansa
and Nandana Rajatheva
Telecommunication Field of Study
School of Engineering
and Technology
Asian Institute of Technology
Thailand
Email: rajath@ait.ac.th

Hlaing Minn
Department of Electrical Engineering
University of Texas at Dallas
Richardson, Tx 75083-0688, U.S.A.
Email: hlaing.minn@utdallas.edu

***Abstract*— In this paper, we derive exact closed form bit error rate (BER) or symbol error rate (SER) expressions for OFDM systems with carrier frequency offset (CFO). We consider the performance of an OFDM system subject to CFO error in frequency flat Rayleigh fading channel with BPSK and 4PSK modulation schemes. Our results can easily be reduced to the respective analytical error rate expressions for the OFDM systems without CFO error. Furthermore, the simulation results are provided to verify the accuracy of the new error rate expressions.**

I. Introduction

Orthogonal frequency division multiplexing (OFDM) is a promising technology for eliminating the frequency selectivity of the wireless communication channels. By definition, OFDM expects the subcarriers to be orthogonal. But, the factors such as carrier frequency mismatching, time variations due to Doppler shift or phase noise usually eliminate the orthogonality of the subcarriers. This gives rise to intercarrier interference (ICI) which degrades the performance of OFDM systems significantly [1].

A careful literature survey reveals two most widely used methods of error performance analysis of OFDM systems. One method is to approximate the ICI as a Gaussian process based on the central limit theorem [2]-[4]. However, in [5], Keller and Hanzo showed that such approximation leads to wrong results (in their own words "pessimistic") for high signal to noise ratios. On the other hand, Pollet *et al* [6] derive analytically the signal to noise ratio degradation due to the ICI. Zhao and Haggman [7] use the moments of ICI distribution to come up a with more accurate BER result. The approach due to Sathananthan and Tellambura [8] is unique in comparison with all the above approaches since they use characteristic functions and Beaulieu series [9] to derive exact BER expressions in the presence of ICI. However, their exact results are given only for the AWGN channel in the form of infinite series expansions.

We follow the procedure presented in Sathananthan and Tellambura with different mathematical insight to come up with exact closed form BER/SER expressions for OFDM with ICI. We show that the ICI terms have a probability density function of a mixture of Gaussian densities with properly selected weighting coefficients. The standard error rate formulae without ICI are shown to be some special cases of our more general results. Furthermore, selected numerical results are provided to verify the accuracy of the formulation.

This paper is organized as follows. Section II derives the exact closed form BER/SER expressions for BPSK and QPSK schemes over AWGN and Rayleigh fading channels. the degenerated solution of the new formulae are also discussed there. Section III provides the simulation results as an alternative verification of the theoritical results given in Section II. Finally, conclusive remarks are made in section IV.

II. Exact Error Rate Expressions with CFO

We assume perfect quasi-static flat fading channel. The sampled signal for the kth subcarrier after going through the fast Fourier transform processor is given by [3]

$$r_k = \alpha X_k S_1 + \alpha \sum_{l=1, l\neq k}^{N} S_{l-k+1} X_l + n_k, \quad k =, 1, 2, .., N \tag{1}$$

where X_k is the transmitted symbol for the kth subcarrier, α denotes the complex fading coefficient which can be modelled as a complex Gaussian variable having zero mean and variance σ_R^2 per dimension, n_k is the additive white Gaussian noise modelled as a zero mean complex Gaussian random variable having variance σ^2 per dimension and N is the number of subcarriers. The ICI coefficients, S_k is given by [3]

$$S_k = \frac{\sin\left(\pi[k-1+\epsilon]\right)}{N\sin\left(\frac{\pi[k-1+\epsilon]}{N}\right)} \exp\left\{j\pi\left(1-\frac{1}{N}\right)(k-1)\right\} \tag{2}$$

where ϵ is the normalized frequency offset. One should note that, (1) becomes an ideal AWGN channel if we put $\alpha = 1$. Without loss of generality, we assume equiprobable message symbols and consider error rates for the first subcarrier. Before starting the main derivation, we present an important identity between product and sum of of cosines as

$$\prod_{k=1}^{M} \cos\left(\phi_k\right) \equiv \frac{1}{2^{M-1}} \sum_{k=1}^{2^{M-1}} \cos\left(\mathbf{\Phi}^T \mathbf{e}_k\right) \tag{3}$$

where $\mathbf{\Phi} = (\phi_1\ \phi_2\ \ldots\ \phi_M)^T$, $\mathbf{e}_k$ is the kth column of a more general $M \times 2^{M-1}$ matrix $\mathbf{E}_M$ and $(\cdot)^T$ denotes the transpose of a matrix. The kth row of $\mathbf{E}_M^T$ is essentially the binary representation of the number $2^M - k$, where zeros are replaced with -1s. The matrix $\mathbf{E}_M$ for the values of $M = 3, 4$ can be written as

$$\mathbf{E}_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$\mathbf{E}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}.$$

Identity (3) can be verified very easily by repeatedly applying the identity $\cos C \cos D \equiv \frac{1}{2}[\cos(C+D) + \cos(C-D)]$ to the product of two cosine terms taken at a time, in the left side.

A. BPSK Modulation

1) AWGN Channel: For BPSK modulation $X_k \in \{1, -1\}$ and we consider the first subcarrier with the transmitted symbol 1. Since the constellation is purely real, only the real part of (1) will be considered. Following [8], the characteristic function (CHF) of the real part of r_1, $\Re(r_1)$ can be written as

$$\phi_{\Re(r_1)}(\omega) = \exp\left\{j\omega\Re\left(S_1\right) - \frac{\omega^2\sigma^2}{2}\right\} \prod_{l=2}^{N} \cos\left(\omega\Re[S_l]\right) \tag{4}$$

and it can further be written using (3) as

$$\phi_{\Re(r_1)}(\omega) = \frac{1}{2^{N-2}} \exp\left\{j\omega\Re\left(S_1\right) - \frac{\omega^2\sigma^2}{2}\right\} \times \sum_{k=1}^{2^{N-2}} \cos\left(\omega\Re\left[\mathbf{S}^T\mathbf{e}_k\right]\right) \tag{5}$$

where $\mathbf{S} = (S_2\ S_3\ \ldots\ S_N)^T$, $\mathbf{E}_{N-1}$ is of dimension $(N-1) \times 2^{N-2}$ and $j = \sqrt{-1}$. Using the Euler's relationship $\cos x = \frac{\exp(jx)+\exp(-jx)}{2}$ and after rearranging the terms we get

$$\phi_{\Re(r_1)}(\omega) = \frac{1}{2^{N-1}} \sum_{k=1}^{2^{N-2}} \left(\exp\left\{j\omega\theta_k - \frac{\omega^2\sigma^2}{2}\right\} + \exp\left\{j\omega\beta_k - \frac{\omega^2\sigma^2}{2}\right\}\right) \tag{6}$$

where $\theta_k = \Re\left(S_1 + \mathbf{S}^T\mathbf{e}_k\right), \beta_k = \Re\left(S_1 - \mathbf{S}^T\mathbf{e}_k\right)$. It is obvious that (6) represents the CHF of a mixture of Gaussian density functions. Since the hypotheses are binary, an error occurs if the $\Re(r_1) < 0$. Now, the probability of a bit error can be written as

$$P_b\left(\xi\right) = \frac{1}{2^{N-1}} \sum_{k=1}^{2^{N-2}} \left\{Q\left(\sqrt{2\gamma}\theta_k\right) + Q\left(\sqrt{2\gamma}\beta_k\right)\right\} \tag{7}$$

where $\gamma = \frac{E_b}{N_0}$, $Q(x)$ is the Gaussian Q function. One should note that in our case of interest $E_b = 1$ and $\sigma^2 = \frac{N_0}{2}$ with N_0 being the noise spectral density. If we let $\epsilon = 0$, then $\theta_k = \beta_k = 1$ and it is easy to see that (7) simplifies to [10,eq. 8.3].

2) Rayleigh Fading Channel: In the Rayleigh case with coherent detection, the conditional bit error probability can be written using (7) as

$$P_b\left(\xi|\alpha\right) = \frac{1}{2^{N-1}} \sum_{k=1}^{2^{N-2}} \left\{Q\left(\sqrt{2\gamma}\theta_k|\alpha|\right) + Q\left(\sqrt{2\gamma}\beta_k|\alpha|\right)\right\} \tag{8}$$

where $|z|$ denotes the absolute value of a complex number z. The bit error probability is given by [10,eq. 8.102]

$$P_b\left(\xi\right) = \int_0^{\infty} P_b\left(\xi|\alpha\right) f\left(|\alpha|\right) d|\alpha| \tag{9}$$

where $|\alpha|$ has a Rayleigh distribution given by

$$f(|\alpha|) = \frac{|\alpha|}{\sigma_R^2} \exp\left(-\frac{|\alpha|^2}{2\sigma_R^2}\right). \tag{10}$$

Using the Craig's formula [11] for Gaussian Q function in (9)

$$Q(x) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\psi}\right) d\psi, \qquad x > 0 \tag{11}$$

and after some algebraic manipulations, (9) gives the bit error probability as

$$P_b(\xi) = \frac{1}{2} - \frac{1}{2^N} \sum_{k=1}^{2^{N-2}} \sqrt{\frac{2\sigma_R^2 \theta_k^2 \gamma}{1 + 2\sigma_R^2 \theta_k^2 \gamma}} + \sqrt{\frac{2\sigma_R^2 \beta_k^2 \gamma}{1 + 2\sigma_R^2 \beta_k^2 \gamma}}. \tag{12}$$

This leads to [10,eq. 8.104] when $\epsilon = 0$.

B. QPSK Modulation

1) AWGN Channel: As usual we consider the first subcarrier with the transmitted symbol $X_1 = 1 + j$ with $X_k = \{\pm 1 \pm j\}$ and we define

$$u = X_k S_1 + \sum_{l=1, l \neq k}^{N} S_{l-k+1} X_l. \tag{13}$$

The probability of a correct decision is the probability that $u + n$ lies inside the first quadrant of the complex plane and it is given in [8] as

$$P(u + n \in D_1 | X_1 = 1 + j, u) = Q\left(\frac{-\Re(u)}{\sigma}\right) \times Q\left(\frac{-\Im(u)}{\sigma}\right) \tag{14}$$

where D_1 is the first quadrant of the complex plane and $\Im(Z)$ is the imaginary part of z. It is natural to integrate (14) with respect to u to find the average correct symbol decision probability. We need to have the pdf of u to perform this integration and the next step is the derivation of the pdf.

Since the variable u is two dimensional and $X_1 = 1 + j$, its two dimensional CHF can be written as [8]

$$\phi(\omega_I, \omega_Q) = e^{(j\omega_I[\Re(S_1) - \Im(S_1)] + j\omega_Q[\Im(S_1) + \Re(S_1)])} \times \prod_{l=2}^{N} \Big\{ \cos(\omega_I \Re(S_l) + \omega_Q \Im(S_l)) \times \cos(\omega_I \Im(S_l) - \omega_Q \Re(S_l)) \Big\}. \tag{15}$$

More compact version of (15) can be written using the vector notations as

$$\phi(\omega_I, \omega_Q) = \exp\left\{ j\mathbf{\Omega}^T \left(\mathbf{S}_A^1 - \mathbf{S}_B^1\right)\right\} \prod_{l=2}^{N} \cos\left(\mathbf{\Omega}^T \mathbf{S}_A^l\right) \cos\left(\mathbf{\Omega}^T \mathbf{S}_B^l\right) \tag{16}$$

where $\mathbf{\Omega} = (\omega_I \; \omega_Q)^T, \mathbf{S}_A^l = (\Re(S_l) \; \Im(S_l))^T$ and $\mathbf{S}_B^l = (\Im(S_l) \; -\Re(S_l))^T$ for all $l = 1, 2, ..., N$. Then we use (3) to express (16) as

$$\phi(\omega_I, \omega_Q) = \frac{1}{2^{2(N-2)}} \exp\left\{ j\mathbf{\Omega}^T \left(\mathbf{S}_A^1 - \mathbf{S}_B^1\right)\right\} \times \sum_{k=1}^{2^{N-2}} \sum_{n=1}^{2^{N-2}} \cos\left(\mathbf{\Omega}^T \mathbf{S}_A \mathbf{e}_k\right) \cos\left(\mathbf{\Omega}^T \mathbf{S}_B \mathbf{e}_n\right) \tag{17}$$

where $\mathbf{S}_A = \left(\mathbf{S}_A^2 \; \mathbf{S}_A^3 \; \ldots \; \mathbf{S}_A^N\right), \mathbf{S}_B = \left(\mathbf{S}_B^2 \; \mathbf{S}_B^3 \; \ldots \; \mathbf{S}_B^N\right)$, $\mathbf{e}_k$ and $\mathbf{e}_n$ are column matrices taken from more general $(N-1) \times 2^{N-2}$ matrix $\mathbf{E}_{N-1}$. By using Euler's relationship with some rearrangements of terms, (17) can be written as

$$\phi(\omega_I, \omega_Q) = \frac{1}{2^{2(N-1)}} \sum_{k=1}^{2^{N-2}} \sum_{n=1}^{2^{N-2}} \Big\{ \exp\left(j\mathbf{\Omega}^T \mathbf{C}_1\right) + \exp\left(j\mathbf{\Omega}^T \mathbf{C}_2\right) + \exp\left(j\mathbf{\Omega}^T \mathbf{C}_3\right) + \exp\left(j\mathbf{\Omega}^T \mathbf{C}_4\right) \Big\} \tag{18}$$

where $\mathbf{C}_1 = \mathbf{S}_A^1 - \mathbf{S}_B^1 + \mathbf{S}_A \mathbf{e}_k + \mathbf{S}_B \mathbf{e}_l, \mathbf{C}_2 = \mathbf{S}_A^1 - \mathbf{S}_B^1 - \mathbf{S}_A \mathbf{e}_k - \mathbf{S}_B \mathbf{e}_l, \mathbf{C}_3 = \mathbf{S}_A^1 - \mathbf{S}_B^1 + \mathbf{S}_A \mathbf{e}_k - \mathbf{S}_B \mathbf{e}_l$ and $\mathbf{C}_4 = \mathbf{S}_A^1 - \mathbf{S}_B^1 - \mathbf{S}_A \mathbf{e}_k + \mathbf{S}_B \mathbf{e}_l$. Thus, the Fourier transform of (18) yields the two dimensional pdf of u as

$$p(\Re(u), \Im(u)) = \frac{1}{2^{2(N-1)}} \sum_{k=1}^{2^{N-2}} \sum_{n=1}^{2^{N-2}} \times \sum_{m=1}^{4} \delta\{\Re(u) - \psi_{kn}[1, m]\} \, \delta\{\Im(u) - \psi_{kn}[2, m]\} \tag{19}$$

where $\psi_{kl}[p, q]$ is the (p, q)th element of the 2×4 matrix $\mathbf{\Psi}$ defined as $\mathbf{\Psi} = (\mathbf{C}_1 \; \mathbf{C}_2 \; \mathbf{C}_3 \; \mathbf{C}_4)$ and $\delta(x)$ is the delta function. Averaging (14) with respect to the pdf given in (19) yields the SER as

$$P_s(\xi) = 1 - \frac{1}{2^{2N-2}} \sum_{k=1}^{2^{N-2}} \sum_{n=1}^{2^{N-2}} \times \sum_{m=1}^{4} Q\left(-\sqrt{2\gamma}\psi_{kn}[1, m]\right) Q\left(-\sqrt{2\gamma}\psi_{kn}[2, m]\right). \tag{20}$$

In the absence of ICI (i.e. $\epsilon = 0$), $\mathbf{\Psi}$ can be expressed as

$$\mathbf{\Psi} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

since $S_k = 0$ for all $k \neq 1$ and $S_1 = 1$. Upon substituting the elements of the new $\mathbf{\Psi}$ matrix in (20), it reduces to [10,eq. 8.20].

2) Rayleigh Fading Channel: The conditional symbol error rate in the case of coherent detection can be expressed as

$$P_s(\xi|\alpha) = 1 - \frac{1}{2^{2N-2}} \sum_{k=1}^{2^{N-2}} \sum_{n=1}^{2^{N-2}} \times \sum_{m=1}^{4} Q\left(-\sqrt{2\gamma}\psi_{kn}[1,m]|\alpha|\right) \times Q\left(-\sqrt{2\gamma}\psi_{kn}[2,m]|\alpha|\right) \quad (21)$$

where the pdf of $|\alpha|$ is given by (10). We encounter integrals of the form $\int_0^\infty Q(\lambda|\alpha|)\, Q(\mu|\alpha|)\, f(|\alpha|)\, d|\alpha|$, $\int_0^\infty Q(\lambda|\alpha|)\, f(|\alpha|)\, d|\alpha|$ in averaging (21) and they can be solved using [10,eq. 4.8], [10,eq. 5.102] and [10,eq. 5.6] to give the SER as (22) at the bottom. Equation (22) can easily be simplified to [10,eq. 8.108] if we let $\epsilon = 0$.

III. Simulation Results and Discussion

In Fig. 1, SER are shown for an OFDM system with $N = 16$ for BPSK modulation scheme. It is obvious that the simulation results agree with those calculated with (4), (5). The SER performance of an OFDM system with QPSK modulation and $N = 8$ is shown in Fig. 2. The accuracy of the newly derived SER expressions (20), (22) for QPSK modulation is thus verified by the simulation results. Careful inspection of the SER/BER formulae reveals that the computational complexity increases exponentially with the number of subcarriers. However, Fig. 3 shows that only a couple of subcarriers introduce the significant interference terms (It should be noted that S_k is a periodic sequence having a period of N). Based on these observations, a compromise can be reached between the computational complexity and mathematical exactness by truncating the number of terms involved in the formulae in calculating the exact results.

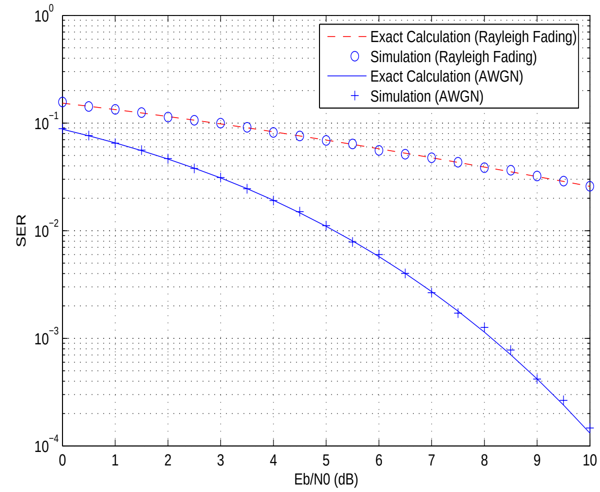

Fig. 1. Probability of symbol error for BPSK with $\varepsilon = 0.1$ and $\sigma_R^2 = 0.5$.

Figure 4 shows the SER performance of an OFDM system having $N = 128$ subcarriers and BPSK/QPSK modulation schemes over frequency flat Rayleigh fading channel, where the number of subcarriers are truncated to 13 in calculating the theoretical BER/SER values. As can be seen, the truncation does not introduce much deviation to the real performance given by simulation results. Furthermore, one can truncate at a larger number of ICI terms to obtain more accurate result at the expense of computational rigour.

IV. Conclusion

In this paper, we derive exact closed form BER/SER expressions for OFDM systems with ICI over AWGN and Rayleigh flat fading channels. The performance of BPSK and QPSK modulation schemes are considered with the new mathematical formulation introduced here. The new formulae can easily be simplified to give the corresponding results available in the literature for ICI free cases. Even though the computational complexity

$$P_s(\xi) = \frac{3}{4} - \frac{1}{2^{2N-1}} \sum_{k=1}^{2^{N-2}} \sum_{n=1}^{2^{N-2}} \sum_{m=1}^{4} \sqrt{\frac{2\sigma_R^2\psi_{kn}^2[1,m]\gamma}{1+2\sigma_R^2\psi_{kn}^2[1,m]\gamma}} \left(1 - \frac{1}{\pi}\arctan\left\{\frac{\psi_{kn}[1,m]}{\psi_{kn}[2,m]}\sqrt{1+\frac{1}{2\sigma_R^2\psi_{kn}^2[1,m]\gamma}}\right\}\right) + \sqrt{\frac{2\sigma_R^2\psi_{kn}^2[2,m]\gamma}{1+2\sigma_R^2\psi_{kn}^2[2,m]\gamma}} \left(1 - \frac{1}{\pi}\arctan\left\{\frac{\psi_{kn}[2,m]}{\psi_{kn}[1,m]}\sqrt{1+\frac{1}{2\sigma_R^2\psi_{kn}^2[2,m]\gamma}}\right\}\right) \quad (22)$$

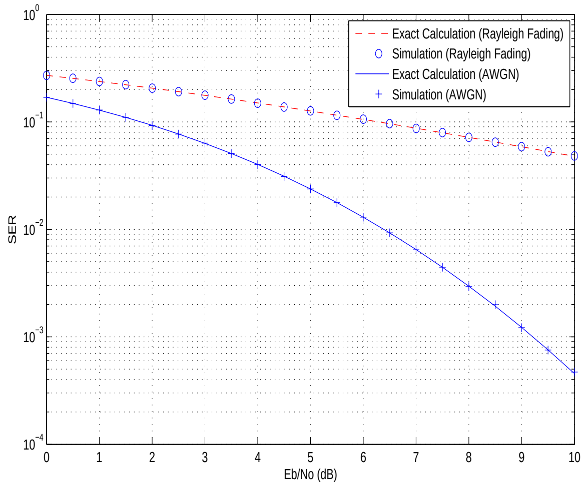

Fig. 2. Probability of symbol error for QPSK with $\varepsilon = 0.1$ and $\sigma_R^2 = 0.5$.

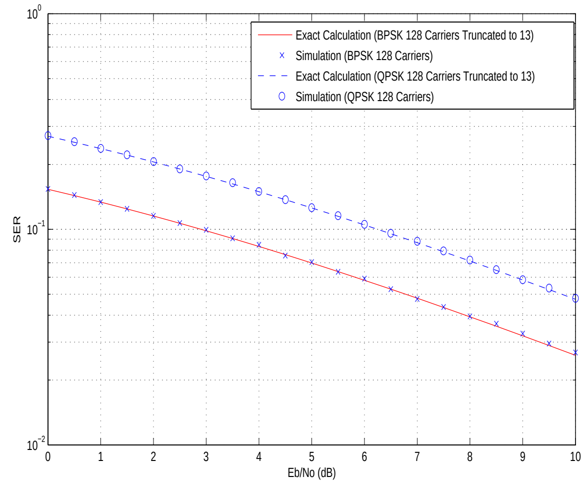

Fig. 4. Probability of symbol error for BPSK/QPSK over Rayleigh fading channel with $\varepsilon = 0.1$ and $\sigma_R^2 = 0.5$.

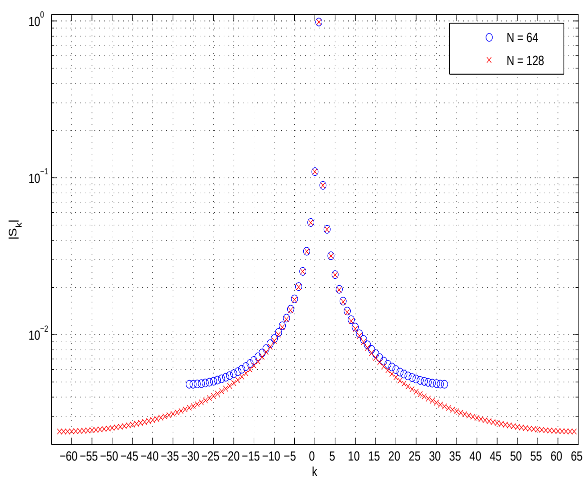

Fig. 3. The amplitudes of the ICI coefficients with $\varepsilon = 0.1$.

of the exact results increases exponentially with the increase of the number of subcarriers, we have shown that the truncation of the ICI sequence will not introduce a significant deviation from the simulation results.

ACKNOWLEDGMENT

The authors would like to thank the government of Finland for the doctoral scholarship provided to the first author. The first author would like to thank Mr. Chaturanga Weeraddhana for being extremely efficient in things getting done with the MATLAB.

REFERENCES

[1] B. Stantchev and G. Fettweis, "Time-varient distortions in OFDM," *IEEE Commun. Lett.*, vol. 4, pp. 312-314, Sep. 2000.

[2] M. Russell and G. L. Stuber, "Interchannel interference analysis for OFDM in a mobile environment ," in *IEEE Vehicular Technology Conf.*, Chicago, IL, Jul. 1995, vol. 2, pp. 820-824.

[3] L. Rugini and P. Banelli, "BER of OFDM systems impaired by carrier frequency offset in multipath fading Channels," *IEEE Trans. Wireless. Commun.*, vol. 4, pp. 2279-2288, Sep. 2005.

[4] K. Zhong, T. T. Tjhung, and F. Adachi, "A general SER formula for an OFDM system with MDPSK in frequency domain over Rayleigh fading channels," *IEEE Trans. Commun.*, vol. 52, pp. 584-594, Apr. 2004.

[5] T. Keller and L. Hanzo, "Adaptive multicarrier modulation: A convenient framework for time-frequency processing in wireless communications," *Proc. IEEE*, vol. 88, pp. 611-640, May 2000.

[6] T. Pollet, M. Van Bladel, and M. Moeneclaey, "BER sensivity of OFDM systems to carrier frequency offset and Wiener phase noise," *IEEE Trans. Commun.*, vol. 43, pp. 191-193, Feb. /Mar. /Apr. 1995.

[7] Y. Zhao and S. G. Haggman, "BER analysis of OFDM communication systems with intercarrier interference," in *IEEE Int. Conf. Communication Technology*, Beijing, China, Oct. 1998, vol. 2, pp. 1-5. S38-02.

[8] K. Sathananthan and C. Tellambura, "Probability of error calculation of OFDM systems with frequency offset," *IEEE Trans. Commun.*, vol. 49, pp. 1884-1888, Nov. 2001.

[9] N. C. Beaulieu, "An infinite series for the computation of the complementary probability distribution function of a sum of independent random variables and its application to the sum of Rayleigh random variables," *IEEE Trans. Commun.*, vol. 38, pp. 1463-1474, Sept. 1990.

[10] M. K. Simon and M. -S. Alouini, *Digital Communication Over Fading Channels*, 2nd ed. New York: Wiley, 2005.

[11] J. W. Craig, "A new, simple and exact result for calculating the probability of errors for two dimensional signal constellations," *IEEE MILCOM'91 Conf. Rec.*, Boston,MA, pp. 25.5.1-25.5.5, May.