# OPTIMAL TRAINING SIGNALS FOR SISO OFDM CHANNEL ESTIMATION IN THE ABSENCE/PRESENCE OF FREQUENCY OFFSETS

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***Abstract*— All existing training signal designs for channel estimation in OFDM systems assume no frequency offsets. In practice, frequency offset is unavoidable and seriously degrades the performance of OFDM systems. In this paper, we address the problem of designing optimal training signals for SISO OFDM channel estimation in the presence of frequency offsets. First, we present optimal training signals for OFDM channel estimation in the absence of frequency offset which include all existing optimal training signals for SISO OFDM channel estimation as a subset. Then we derive the optimal training signals for SISO OFDM channel estimation which are the most robust to frequency offsets. In the absence of frequency offsets, channel estimation mean square error depends only on the ratio of training signal energy to noise variance. In the presence of frequency offsets, channel estimation mean square error depends on this ratio as well as on the channel power delay profile and the frequency offset. Analytical and simulation results show that the performance improvement achieved by the proposed optimal training signals can be quite significant for moderate-to-high values of SNR and frequency offsets.**

## I. INTRODUCTION

Training signal design for OFDM channel estimation has attracted significant research attention. For SISO OFDM systems, the optimal training sequences and pilot tones for frequency-selective block-fading channel estimation were investigated in [1][2] and [3]. For MIMO OFDM systems, [4] described an optimal training signal design for frequency-selective block-fading channel estimation in pilot-data-multiplexed scheme while [5] presented an optimal training signal design for the scheme where all sub-carriers are used as pilot tones. Recently, [6] presented general optimal training signal designs for frequency-selective block-fading channel estimation in MIMO OFDM systems which include all existing optimal training signal designs in the literature as special cases.

To the best of our knowledge, all existing training signal designs for OFDM channel estimation assume no frequency offset. In practice, frequency offset is unavoidable due to local oscillator mismatches and Doppler shifts of the mobile wireless channels. Frequency offset causes a loss of orthogonality among the sub-carriers which in turn seriously degrades the performance of OFDM systems. To mitigate OFDM's high sensitivity to frequency offsets, several highly-accurate frequency offset estimators were proposed. However, there will still be some residual frequency offset after applying these frequency offset compensation techniques. It is unclear how the existing optimal training signals behave in the presence of some (residual) frequency offsets. Hence, in this paper, we investigate optimal training signals for SISO OFDM channel estimation in the presence of frequency offsets. We first present optimal training signals for OFDM channel estimation in the absence of frequency offset which include all existing optimal training signals for SISO OFDM channel estimation as a subset. Then we derive the optimal training signals for SISO OFDM channel estimation which are the most robust to frequency offsets. Our results show that the performance improvement of the proposed optimal training signals can be quite significant for moderate-to-high values of SNR and frequency offsets.

The rest of this paper is organized as follows. In Section II, the signal model and the optimality condition are described. Section III presents optimal training signals in the absence of frequency offset. Section IV derives the optimal training signals in the presence of frequency offsets. Numerical examples, simulation results and discussions are presented in Section V and the paper is concluded in Section VI.

## II. SIGNAL MODEL AND OPTIMALITY CRITERION

Consider an OFDM system with $K$ sub-carriers and $N_g$ cyclic prefix samples. For channel estimation, $Q$ ($\geq 1$) OFDM training symbols are transmitted. The channel impulse response (CIR) (including all transmit/receive filtering effects) is assumed to have $L$ taps (where $L \leq N_g + 1$) and is quasi-static over $Q$ OFDM symbol intervals.

Let $\boldsymbol{C}_q = [c_q[0], \ldots, c_q[K-1]]^T$ be the pilot tones vector at the $q$-th symbol interval where the superscript $T$ denotes the transpose. Furthermore, let $\{s_q[k] : k = -N_g, \ldots, K-1\}$ be the corresponding time-domain complex baseband training samples, including the cyclic prefix samples. Define $\boldsymbol{S}[q]$ as the training signal matrix of size $K \times L$ at the $q$-th symbol interval whose elements are given by $[\boldsymbol{S}[q]]_{m,l} = s_q[l-m]$ for $m \in \{0, \ldots, K-1\}$ and $l \in \{0, \ldots, L-1\}$.

Define $\boldsymbol{s}_q$ as the 0-th column of $\boldsymbol{S}[q]$. Then, the $l$-th column of $\boldsymbol{S}[q]$ is the $l$-sample cyclically-shifted version of $\boldsymbol{s}_q$ denoted by $\boldsymbol{s}_q^{((l))}$. Assume that $K = ML_0$ where $M \in \{1, 2, \ldots\}$, and $L_0 \geq L$ and let $\boldsymbol{h}$ denote the length-$L$ CIR vector. After cyclic prefix removal at the receiver, denote the received vector of length $K$ at the $q$-th symbol interval by $\boldsymbol{r}_q$. Then, the received vector over the $Q$ symbols interval is given by

$$\boldsymbol{r} = \boldsymbol{S}\boldsymbol{h} + \boldsymbol{n} \quad (1)$$

$$\text{where } \boldsymbol{r} = [\boldsymbol{r}_0^T, \boldsymbol{r}_1^T, \ldots, \boldsymbol{r}_{Q-1}^T]^T \quad (2)$$

$$\boldsymbol{S} = \left[\boldsymbol{S}^T[0], \boldsymbol{S}^T[1], \ldots, \boldsymbol{S}^T[Q-1]\right]^T \quad (3)$$

and $\boldsymbol{n}$ is a length-$KQ$ vector of zero-mean, circularly-symmetric, uncorrelated complex Gaussian noise samples with equal variance of $\sigma_n^2$.

The least square channel estimate (also maximum likelihood), assuming $\boldsymbol{S}^H\boldsymbol{S}$ has full rank, is given by [7]

$$\hat{\boldsymbol{h}} = (\boldsymbol{S}^H\boldsymbol{S})^{-1}\boldsymbol{S}^H\boldsymbol{r} \quad (4)$$

and the corresponding mean square error (MSE) is given by $\sigma_n^2 \, tr\{(\boldsymbol{S}^H\boldsymbol{S})^{-1}\}$ where the superscript $H$ denotes the Hermitian transpose. The minimum MSE is achieved when

$$\boldsymbol{S}^H\boldsymbol{S} = E_t\boldsymbol{I} \quad (5)$$

$$\text{where} \quad E_t = \sum_{q=0}^{Q-1}\sum_{k=0}^{K-1}|s_q[k]|^2. \quad (6)$$

The corresponding minimum MSE is $L\sigma_n^2/E_t$. Condition (5) can be equivalently stated as

$$\sum_{q=0}^{Q-1}\boldsymbol{S}^H[q]\boldsymbol{S}[q] = E_t\boldsymbol{I}. \quad (7)$$

## III. Optimal Training Signals in the Absence of Frequency Offset

For completeness, in the following we summarize the main DFT properties used in this paper. Let $X[n] = \sum_{k=0}^{K-1} x[k]e^{-j2\pi kn/K}$ and $x[k] = \frac{1}{K}\sum_{n=0}^{K-1} X[n]e^{j2\pi kn/K}$, i.e. $X[n] \overset{\mathcal{F}}{\longleftrightarrow} x[k]$.

Property-1: For any $K$, if $X[n] = a$, $\forall n$, where $a \in \mathbb{C}$ and $\mathbb{C}$ is the field of complex numbers, then $x[k] = a\delta[k]$ where $\delta[k]$ is a discrete unit impulse function, and vice versa.

Property-2: Assume that $K = ML_0$ for $M$=1, 2, .... If

$$X[n] = \begin{cases} a, & n = mM; m = 0,\ldots,L_0-1; a \in \mathbb{C} \\ 0, & \text{elsewhere,} \end{cases} \quad (8)$$

then

$$x[k] = \begin{cases} aL_0/K, & k = nL_0; n = 0,\ldots,M-1 \\ 0, & \text{elsewhere,} \end{cases} \quad (9)$$

and vice versa.

Property-3:

$$X[(n-l)_K] \overset{\mathcal{F}}{\longleftrightarrow} e^{j2\pi lk/K}x[k]$$

where $(\cdot)_K$ denotes modulo-$K$ operation, hence representing a cyclically-shifted version. Its dual form is given by

$$x[(k-m)_K] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j2\pi mn/K}X[n].$$

This section considers training design for channel estimation based on observation over $Q$ symbols interval. Let us consider the condition in (7). The following are observed:

(A-i) The full rank condition in (7) means that there must be at least $L$ different (nonzero) tones of one symbol duration over $Q$ symbols interval.

(A-ii) The condition in (7) gives the following

$$\sum_{q=0}^{Q-1}(\boldsymbol{s}_q^{((l))})^H\boldsymbol{s}_q^{((m))} = 0, \ \forall l \neq m; \ l,m \in \{0,1,\ldots,L-1\} \quad (12)$$

which means that

$$\sum_{q=0}^{Q-1}\sum_{n=0}^{K-1}|c_q[n]|^2e^{j2\pi dn/K} = 0 \text{ for } d = \pm1,\pm2,\ldots,\pm(L-1). \quad (13)$$

By defining

$$E[n] = \sum_{q=0}^{Q-1}|c_q[n]|^2, n = 0,\ldots,K-1, \quad (14)$$

we can express (13) as

$$\sum_{n=0}^{K-1}E[n]e^{j2\pi dn/K} = 0 \text{ for } d = \pm1,\pm2,\ldots,\pm(L-1). \quad (15)$$

Note that $E_t = \sum_{n=0}^{K-1}E[n]$. Using Property-2 and 3, we obtain the following condition satisfying (13) for $K = ML_0$, $L_0 \geq L$:

$$E[n] = \sum_{l=0}^{M-1}\sum_{m=0}^{L_0-1}a_l\,\delta[n-l-mM], \ , \ a_l \geq 0 \quad (16)$$

$$\text{or } E[n] = \sum_{k=0}^{V-1}\sum_{m=0}^{L_0-1}a_k\,\delta[n-l_k-mM], \ a_k > 0 \quad (17)$$

where $V \in \{1,\ldots,M\}$, $l_k \in \{0,\ 1,\ \ldots,\ M-1\}$ and $l_k \neq l_i$ if $k \neq i$. Note that $L_0\sum_l a_l = E_t$ and hence, at least one $a_l$ must be nonzero for nonzero $E_t$. The energies of non-zero pilot tones within a symbol duration may not necessarily be the same unless $Q = 1$ and only $L_0$ pilot tones are used. For $Q = 1$, $K = ML_0$ and $L_0 \geq L$, (16) or (17) gives the optimal pilot tones in the absence of frequency offset as

$$c^\star[n] = \sum_{l=0}^{M-1}\sum_{m=0}^{L_0-1}A_l\,\alpha_{mM+l}\,\delta[n-l-mM] \quad (18)$$

$$\sum_{l=0}^{M-1}A_l^2 = KE_t/L_0, \ A_l \geq 0, \ |\alpha_i| = 1. \quad (19)$$

In general, for $Q \geq 1$, $K = ML_0$ and $L_0 \geq L$, the optimal pilot tones in the absence of frequency offset are given by

$$c_q^\star[n] = \sum_{l=0}^{M-1}\sum_{m=0}^{L_0-1}A_{q,m,l}\,\alpha_{qK+mM+l}\,\delta[n-l-mM] \quad (20)$$

$$\sum_{q=0}^{Q-1}\sum_{l=0}^{M-1}\sum_{m=0}^{L_0-1}A_{q,m,l}^2 = KE_t \quad (21)$$

$$\sum_{q=0}^{Q-1}A_{q,m_1,l}^2 = \sum_{q=0}^{Q-1}A_{q,m_2,l}^2, \ \ A_{q,m,l} \geq 0, \ |\alpha_i| = 1. \quad (22)$$

For $Q = 1$, the optimal training signals defined by (17) with $V = 1$ are equi-spaced $L_0$ pilot tones with equi-energy (where $L_0 \geq L$, and $K = ML_0$) which were the optimal pilot tones presented in [1] [2]. If $L_0 = L$ is imposed, the optimal pilot tones from [3] are obtained. For $Q = 1$ and $V > 1$, the optimal pilot tones from (17) are composed of $V$ disjoint sets of equi-spaced, equi-energy $L_0$ pilot tones. Pilot energies for different sets can be different and there is no restriction on the spacing between any two disjoint sets. In other words, all $VL_0$ pilot tones need not be equi-spaced, nor of equal-energy. For $Q > 1$, the requirements on the pilot tones are just spread out over $Q$ symbols interval.

As an illustrative example, Table I lists some representatives of optimal pilot tones for $Q = 1$ in the absence of frequency offsets. $\{\alpha_i\}$ are unit amplitude symbols. The first three pilot tone vectors correspond to the existing optimal pilot tones from [3][1][2]. The remaining pilot tone vectors are new results. Table II lists optimal pilot tones for $Q = 2$ in the absence of frequency offsets.

Note that for $L_1 > L_0 \geq L$ and a positive integer $U$, if $K = L_0 L_1 U$ and there are disjoint sets of eqi-spaced, equi-energy $L_0$ and $L_1$ pilot tones then (15) will give additional optimal pilot tones not covered by (17). These additional optimal pilot tones are composed of disjoint sets of $L_0$ and $L_1$ tones. Within each set of $L_0$ equi-energy pilot tones, the spacing is $K/L_0$ while within the set with $L_1$ equi-energy pilot tones, the spacing is $K/L_1$. Pilot energies for different sets can be different. An example is given in Table III. This type of optimal pilot tones can be extended for $L \leq L_0 < L_1 < L_2 < \ldots < L_d$ as long as $K = U \prod_{i=0}^{d} L_i$ and there exist disjoint sets of equi-spaced $L_i$ pilot tones with spacing $K/L_i$ for all $i$. But for practical systems where $K$ is a power of 2, the optimal pilot type described above is not likely to exist and those defined by (17) would cover all optimal pilot tones.

## IV. The Optimal Training Signals in the Presence of Frequency Offsets

In practical systems where frequency offsets are unavoidable, the optimal training signals presented in the previous section may not result in the same MSE performance. This section derives training signals that are optimal in terms of their robustness to frequency offsets. In the presence of a normalized (by the sub-carrier spacing) carrier frequency offset $v$, the received vector in (2) becomes

$$\boldsymbol{r} = \boldsymbol{W}(v)\, \boldsymbol{S}\boldsymbol{h} + \boldsymbol{n} \tag{23}$$

where $\boldsymbol{W}(v) = diag\{\boldsymbol{W}_0(v)\ ,\ e^{j2\pi v(K+N_g)/K}\boldsymbol{W}_0(v),\ \ldots,\ e^{j2\pi v(Q-1)(K+N_g)/K}\boldsymbol{W}_0(v)\}$, $\boldsymbol{W}_0(v) = diag\{1,\ e^{j2\pi v/K},\ e^{j2\pi 2v/K},\ \ldots,\ e^{j2\pi(K-1)v/K}\}$. The corresponding channel estimate obtained using an optimal training signal from the previous section is

$$\hat{\boldsymbol{h}} = \frac{1}{E_t}\boldsymbol{S}^H\boldsymbol{r} \tag{24}$$

$$= \boldsymbol{h} - \frac{1}{E_t}\{\boldsymbol{S}^H(\boldsymbol{I} - \boldsymbol{W}(v))\boldsymbol{S}\boldsymbol{h} - \boldsymbol{S}^H\boldsymbol{n}\} \equiv \boldsymbol{h} - \Delta_{\boldsymbol{h}}. \tag{25}$$

We define the normalized MSE as

$$\mathrm{NMSE} = \frac{\mathrm{MSE}}{L} = \frac{\mathrm{E}[\Delta_{\boldsymbol{h}}{}^{\mathrm{H}}\Delta_{\boldsymbol{h}}]}{L}$$

$$= \frac{\sigma_n^2}{E_t} + \frac{\mathrm{Tr}[\boldsymbol{S}^H(\boldsymbol{I} - \boldsymbol{W}(v))\boldsymbol{S}\boldsymbol{C}_{\boldsymbol{h}}\boldsymbol{S}^H(\boldsymbol{I} - \boldsymbol{W}(v))^H\boldsymbol{S}]}{LE_t^2} \tag{26}$$

$$\equiv \mathrm{NMSE}_0 + \Delta_{\mathrm{NMSE}}. \tag{27}$$

In (26) , the first term is the NMSE obtained without any frequency offset and the second term is the extra NMSE caused by the frequency offset $v$.

We will investigate which training signals are the best (most robust to frequency offsets) among the optimal training signals presented in the previous section. Equivalently, we will find the best training signal matrices $\boldsymbol{S}^\star$ which give the minimum extra MSE, i.e.,

$$\boldsymbol{S}^\star = \underset{\boldsymbol{S}}{arg\ min}\ \mathrm{Tr}[\boldsymbol{S}^H\boldsymbol{V}(v)\boldsymbol{S}\boldsymbol{C}_{\boldsymbol{h}}\boldsymbol{S}^H\boldsymbol{V}(v)^H\boldsymbol{S}] \tag{28}$$

where $\boldsymbol{V}(v) \equiv (\boldsymbol{I} - \boldsymbol{W}(v))$ and $\boldsymbol{S}$ is constrained to be circulant as described in Section II.

Since $\boldsymbol{S}^H\boldsymbol{V}(v)\boldsymbol{S}\boldsymbol{C}_{\boldsymbol{h}}\boldsymbol{S}^H\boldsymbol{V}(v)^H\boldsymbol{S}$ is in the form of $\boldsymbol{G}\boldsymbol{G}^H$, it is a Hermitian positive semi-definite matrix. Let $\boldsymbol{X} = \boldsymbol{G}\boldsymbol{G}^H = \boldsymbol{S}^H\boldsymbol{V}(v)\boldsymbol{S}\boldsymbol{C}_{\boldsymbol{h}}\boldsymbol{S}^H\boldsymbol{V}(v)^H\boldsymbol{S}$. Define $\boldsymbol{Y}$=$\boldsymbol{X} + \boldsymbol{\Lambda}_{\boldsymbol{X}_d} + b\boldsymbol{I}$ where $\boldsymbol{X}_d$ is a diagonal $\boldsymbol{X}$, $\boldsymbol{X}_d + \boldsymbol{\Lambda}_{\boldsymbol{X}_d} = a\boldsymbol{I}$, and $a, b > 0$. Consider groups of $\boldsymbol{Y}$ with the same determinant $(a+b)^L$. Using the arithmetic-geometric mean inequality [1], we conclude that Tr$[\boldsymbol{Y}]$ and hence Tr$[\boldsymbol{X}]$ will be minimum for $\boldsymbol{X} = \boldsymbol{X}_d$ within each group. Hence, we just need to consider diagonal matrices $\boldsymbol{X}_d$. We assume that the channel correlation matrix is given by $\boldsymbol{C}_{\boldsymbol{h}}$ = diag$\{\sigma_0^2,\ \sigma_1^2,\ \ldots,\ \sigma_{L-1}^2\}$. Since $\boldsymbol{C}_h$ is diagonal, $\boldsymbol{X}$ will also be diagonal when $\boldsymbol{S}^H\boldsymbol{V}(v)\boldsymbol{S}$ is diagonal. Since $\boldsymbol{S}$ is circulant, $\boldsymbol{S}^H\boldsymbol{V}(v)\boldsymbol{S}$ will be diagonal if and only if

$$s_q[n] = \sum_{i=0}^{d_q-1} A_{q,i}\ \delta[n - l_{q,i}],\ \ l_{q,i+1} - l_{q,i} \geq L,\ \forall i, q \tag{29}$$

where $0 \leq l_{q,i} \leq K-1$. The $k$-th diagonal element of the diagonal matrix $\boldsymbol{S}^H\boldsymbol{V}(v)\boldsymbol{S}$ is then given by $\sum_{q=0}^{Q-1}\sum_{i=0}^{d_q-1}|A_{q,i}|^2 V_{qK+l_{q,i}+k}$ where $k = 0, \ldots, L-1$ and $V_i$ denotes the $i$-th diagonal element of $\boldsymbol{V}(v)$. Now, we have a diagonal matrix $\boldsymbol{X}_d$ with

$$\mathrm{Tr}[\boldsymbol{X}_d] = \sum_{k=0}^{L-1}\sigma_k^2\left|\sum_{q=0}^{Q-1}\sum_{i=0}^{d_q-1}|A_{q,i}|^2 V_{qK+l_{q,i}+k}\right|^2. \tag{30}$$

Therefore, the best signal matrix $\boldsymbol{S}^\star$ is determined by $\{|A_{q,i}|^2,\ l_{q,i}\}^\star$ where

$$\{|A_{q,i}|^2,\ l_{q,i}\}^\star = \underset{\{|A_{q,i}|^2,\ l_{q,i}\}}{arg\ min} \sum_{k=0}^{L-1}\sigma_k^2\left|\sum_{q=0}^{Q-1}\sum_{i=0}^{d_q-1}|A_{q,i}|^2 V_{qK+l_{q,i}+k}\right|^2 \tag{31}$$

$$\text{subject to } \sum_{q=0}^{Q-1}\sum_{i=0}^{d_q-1}|A_{q,i}|^2 = E_t,$$

$$l_{q,i+1} - l_{q,i} \geq L,\ \text{and}\ 0 \leq l_{q,i} \leq K-1, \forall i, q. \tag{32}$$

Since $\{V_l\}$ depend on $v$, there is no single $\boldsymbol{S}^\star$ which remains optimum for all $v$ and $\{\sigma_i^2\}$.

In practical systems, frequency offset estimation and compensation are typically performed before channel estimation. Hence, during channel estimation, the residual frequency offset is usually very small (i.e., very small $v$ in our signal model). In this case, we can find the optimum $\boldsymbol{S}$ by solving (31). The real and imaginary parts of $\{V_l\}$ for different values of $v$ are plotted in Figure 1 for an OFDM system with $K = 64$, $N_g = 16$ as in the IEEE 802.11a standard.

For very small values of $v$, we have

$$V_l = 1 - e^{j2\pi k_l v/K} \simeq -j2\pi k_l v/K \tag{33}$$

where

$$k_l = \lfloor l/K \rfloor N_g + l. \tag{34}$$

Then, it is straightforward to see from (31) that the best set of $\{|A_{q,i}|^2,\ l_{q,i}\}$ is given by

$$\{|A_{q,i}|^2\}^\star = \begin{cases} E_t & \text{, if } i = 0 \text{ and } q = 0 \\ 0 & \text{, otherwise} \end{cases} \tag{35}$$

$$l_{0,0}^\star = 0. \tag{36}$$

[1] For positive numbers $\lambda_i$, $\prod_{i=1}^{N}\lambda_i \leq (\frac{1}{N}\sum_{i=1}^{N}\lambda_i)^N$ and the equality holds if and only if all $\lambda_i$ are equal.

The corresponding minimum extra NMSE is

$$(\Delta_{\mathrm{NMSE}})_{\min} = \frac{1}{L}\sum_{l=1}^{L-1}\sigma_l^2|V_l|^2 \simeq \frac{1}{L}(2\pi v/K)^2\sum_{l=1}^{L-1}\sigma_l^2 k_l^2 \qquad (37)$$

where $k_l$ is given by (34). The optimum training signal is then given by

$$s_q^\star[n] = A\delta[n]\delta[q],\ |A|^2 = E_t. \qquad (38)$$

Equation (38) together with DFT Property-1 imply that among the optimal pilot tones presented in the previous section, the most robust to frequency offsets is given by

$$c_q^\star[k] = a\ \delta[q],\ a \in \{\mathbb{C}\setminus 0\}, k = 0, 1, \ldots, K-1. \qquad (39)$$

Hence, transmitting the same pilot tone symbol on all sub-carriers gives the optimal channel estimation performance in the presence of frequency offsets. The above result implies that under the same training signal energy constraint, using all sub-carriers as pilot tones and using one training symbol ($Q = 1$) for channel estimation is more robust to frequency offsets.

## V. Discussions and Simulation Results

We numerically evaluated the extra NMSEs for all training signals which were optimal in the absence of frequency offset and confirmed the minimum extra NMSE of the optimal training signals presented in Section IV. For the convenience of numerical evaluation and presentation, $K = 8$ and $L = 2$ will be assumed unless stated otherwise. The range of the extra NMSE is $[1.0297\times10^{-5}, 1.2439\times10^{-3}]$ at $v = 0.01$ and $[1.0292\times10^{-3}, 1.2180\times10^{-1}]$ at $v = 0.1$. The minimum values are achieved by the proposed optimal training signals. One can notice that the extra NMSE is approximately proportional to $v^2$, as also evident from (37).

Some representative examples of the optimal pilot tone vectors in the absence of frequency offset are presented in Tables I-III and those in the presence of frequency offset are given in Table IV.

The minimum NMSEs achieved with the proposed optimal training signals for OFDM systems with $K = 64$, $N_g = 16$, and $Q = 1$ in a multipath channel with $L = 8$ taps and an exponential power delay profile with a 3 dB per tap decaying factor are plotted in Figure 2 for different values of $v$ and SNR $(= E_t/(QK\sigma_n^2))$. At high SNR, $v = 0.1$ introduces a considerable degradation in channel estimation while $v = 0.01$ or less causes insignificant degradation.

For validation purposes, we simulated the NMSE performance in the presence of frequency offsets for the proposed optimal training signal and other training signals which were optimal in the absence of frequency offsets. An OFDM system with $K = 64$, $L = 8$, $Q = 1$ and residual normalized frequency offsets of 0.001, 0.01 and 0.1 are considered. The results are presented in Figure 3 where training#1 is the optimal training signal, training#2 uses $L$ equi-energy, equi-spaced pilot tones (a pseudo-noise (PN) sequence), training#3 consists of $L$ equi-energy, equi-spaced pilot tones with the same pilot-symbol, and training#4 utilizes an equi-energy PN sequence over all sub-carriers. The advantage of the proposed optimal training signals becomes more significant for larger values of frequency offsets (see the results for $v = 0.01$ and $v = 0.1$ in Figure 3). The simulation results agree with the theoretical results and our previous discussions.

TABLE II

Optimal pilot tone vectors in the absence of frequency offset for an OFDM system with $K = 8, L = L_0 = 2$, and $Q = 2$

| Sub-carrier Index | Symbol Index | |
|---|---|---|
| | $q = 0$ | $q = 1$ |
| 0 | $A_{0,0}\alpha_0$ | $\sqrt{A_0^2 - A_{0,0}^2}\alpha_8$ |
| 1 | $A_{1,0}\alpha_1$ | $\sqrt{A_1^2 - A_{1,0}^2}\alpha_9$ |
| 2 | $A_{2,0}\alpha_2$ | $\sqrt{A_2^2 - A_{2,0}^2}\alpha_{10}$ |
| 3 | $A_{3,0}\alpha_3$ | $\sqrt{A_3^2 - A_{3,0}^2}\alpha_{11}$ |
| 4 | $A_{0,1}\alpha_4$ | $\sqrt{A_0^2 - A_{0,1}^2}\alpha_{12}$ |
| 5 | $A_{1,1}\alpha_5$ | $\sqrt{A_1^2 - A_{1,1}^2}\alpha_{13}$ |
| 6 | $A_{2,1}\alpha_6$ | $\sqrt{A_2^2 - A_{2,1}^2}\alpha_{14}$ |
| 7 | $A_{3,1}\alpha_7$ | $\sqrt{A_3^2 - A_{3,1}^2}\alpha_{15}$ |
| $L_0\sum_{l=0}^{M-1}A_l^2 = KE_t, A_k \geq A_{k,i} \geq 0,\ \lvert\alpha_i\rvert = 1$ | | |

TABLE III

Examples of another type of optimal pilot tone vectors in the absence of frequency offset for an OFDM system with $K = 12, L = L_0 = 2, L_1 = 3$ and $Q = 1$

| Sub-carrier Index | | | |
|---|---|---|---|
| 0 | $A_0\alpha_0$ | $A_0\alpha_0$ | $A_0\alpha_0$ |
| 1 | $A_1\alpha_1$ | $A_1\alpha_1$ | $A_1\alpha_1$ |
| 2 | 0 | 0 | $A_2\alpha_2$ |
| 3 | 0 | 0 | 0 |
| 4 | 0 | $A_2\alpha_5$ | $A_0\alpha_3$ |
| 5 | $A_1\alpha_2$ | $A_1\alpha_2$ | 0 |
| 6 | $A_0\alpha_3$ | $A_0\alpha_3$ | $A_2\alpha_4$ |
| 7 | 0 | 0 | $A_1\alpha_5$ |
| 8 | 0 | 0 | $A_0\alpha_6$ |
| 9 | $A_1\alpha_4$ | $A_1\alpha_4$ | $A_2\alpha_7$ |
| 10 | 0 | $A_2\alpha_6$ | 0 |
| 11 | 0 | 0 | 0 |
| $A_i > 0,\ \lvert\alpha_i\rvert = 1$ | | | |

## VI. Conclusions

In this paper, we presented a larger set of optimal training signals than the existing ones in literature for OFDM channel estimation in the absence of frequency offsets. Then we derived optimal training signals for OFDM channel estimation in presence of frequency offsets. We showed in this paper that the most robust optimal pilot tone vectors for SISO OFDM channel estimation in the presence of frequency offsets are the ones that have the same pilot symbol on all sub-carriers. Using only one OFDM training symbol is more robust to frequency offsets than using more than one training symbols. The reduction in the frequency-offset-induced extra MSE achieved by the optimal training signals can be quite significant (one or two orders of magnitude) at moderate and high values of SNR and frequency offsets. Further investigation into the optimal training signal design for MIMO OFDM in the presence of frequency offsets and peak factor limit is underway.

TABLE I

OPTIMAL PILOT TONE VECTORS IN THE ABSENCE OF FREQUENCY OFFSET FOR AN OFDM SYSTEM WITH $K = 8, L = L_0 = 2$, AND $Q = 1$

| Sub-carrier Index | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $A_0\alpha_0$ | 0 | 0 | 0 | $A_0\alpha_1$ | 0 | 0 | 0 |
| $A_0\alpha_0$ | 0 | $A_0\alpha_1$ | 0 | $A_0\alpha_2$ | 0 | $A_0\alpha_3$ | 0 |
| $A_0\alpha_0$ | $A_0\alpha_1$ | $A_0\alpha_2$ | $A_0\alpha_3$ | $A_0\alpha_4$ | $A_0\alpha_5$ | $A_0\alpha_6$ | $A_0\alpha_7$ |
| $A_0\alpha_0$ | 0 | $A_1\alpha_1$ | 0 | $A_0\alpha_2$ | 0 | $A_1\alpha_3$ | 0 |
| $A_1\alpha_0$ | $A_2\alpha_1$ | 0 | 0 | $A_1\alpha_2$ | $A_2\alpha_3$ | 0 | 0 |
| $A_1\alpha_0$ | $A_2\alpha_1$ | $A_3\alpha_2$ | $A_4\alpha_3$ | $A_1\alpha_4$ | $A_2\alpha_5$ | $A_3\alpha_6$ | $A_4\alpha_7$ |
| For each pilot tone vector, $L_0 \sum_i A_i^2 = KE_t,\ A_i > 0,\ \lvert\alpha_i\rvert = 1$ | | | | | | | |

TABLE IV

OPTIMAL PILOT TONE VECTORS IN THE PRESENCE OF FREQUENCY OFFSET FOR AN OFDM SYSTEM WITH $K = 8, L = 2$

| Sub-carrier Index | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $A\alpha$ | $A\alpha$ | $A\alpha$ | $A\alpha$ | $A\alpha$ | $A\alpha$ | $A\alpha$ | $A\alpha$ |
| $A^2 = E_t,\ \lvert\alpha\rvert = 1$ | | | | | | | |

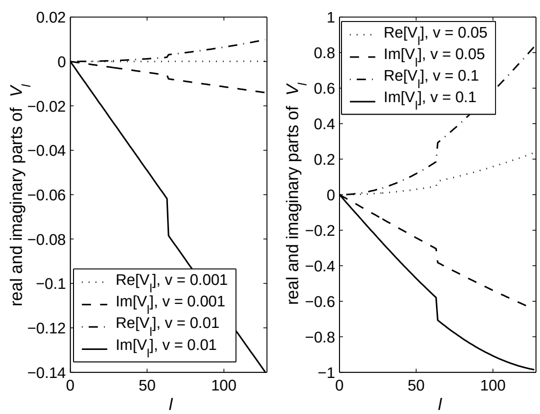


Fig. 1. Real and imaginary parts of $V_l$ for different values of $v$ in OFDM systems with $K = 64$, $N_g = 16$ in an 8-tap multipath Rayleigh fading channel with an exponential power delay profile

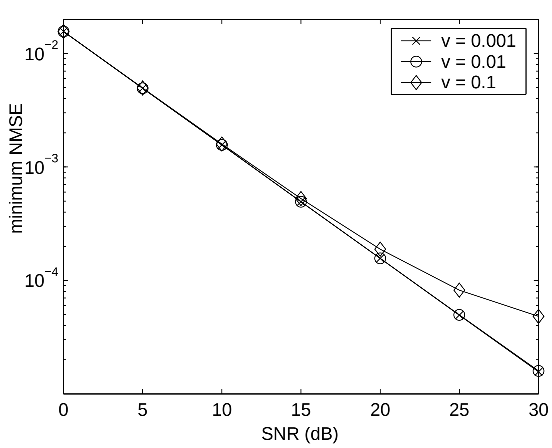


Fig. 2. The minimum NMSE for different values of $v$ and SNR for an OFDM system with $K = 64$, $N_g = 16$ in an 8-tap multipath Rayleigh fading channel with an exponential power delay profile

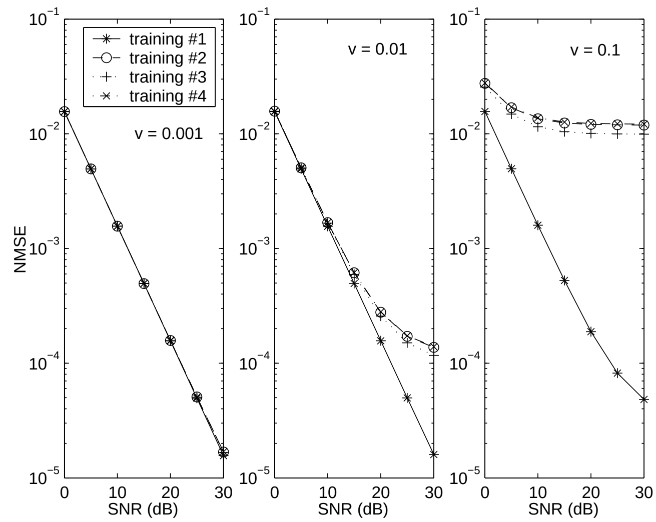


Fig. 3. The NMSE comparison of several training signals for an OFDM system with $K = 64$, $N_g = 16$ in an 8-tap multipath Rayleigh fading channel with an exponential power delay profile

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