THE JOURNAL OF FINANCE • VOL. LII, NO. 5 • DECEMBER 1997

Endogenous Borrowing Constraints With Incomplete Markets

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ABSTRACT

This article develops ways to endogenize the borrowing constraints used in a class of computable incomplete markets models. We allow the constraints to depend on an investor's characteristics such as time preference, risk aversion, and income streams. The proposed constraint can be interpreted as a borrowing limit within which an investor has no incentive to default. Using a numerical algorithm, we find that for an array of structural parameters, the endogenous borrowing constraints can be much less stringent than the ad hoc borrowing constraints adopted by the existing studies.

CURRENT RESEARCH ON ASSET markets reveals that introducing market frictions, such as incomplete markets, transaction costs, short sale constraints, and borrowing constraints, can help resolve the equity premium and risk-free rate puzzles and create pricing kernels with higher volatility (Mankiw (1986), Scheinkman and Weiss (1986), Kahn (1990), Aiyagari and Gertler (1991), Telmer (1993), Lucas (1994), and Heaton and Lucas (1996)). Borrowing constraints, in particular, have been popular in studying the asset pricing implications of a class of computable incomplete markets models for many reasons. For instance, it is widely observed that investors face credit limits in reality because of imperfect information and/or incomplete markets. Imposing borrowing constraints in our economic models can thus capture an important feature of reality. More important, borrowing constraints are typically needed to rule out default and Ponzi schemes, and to ensure the existence of equilibrium for incomplete markets economies. However, the borrowing constraints used in the literature are often specified arbitrarily outside economic models. The purpose of this study is to develop ways to endogenize borrowing constraints in order to provide economic guidance and/or insights into what might drive these constraints and why asset return data seem to imply that they are restrictive.

A number of authors have studied a class of computable incomplete market economies with ad hoc borrowing constraints (Aiyagari and Gertler (1991), Telmer (1993), Lucas (1994), and Heaton and Lucas (1996)). The consensus is

* Carnegie Mellon University. I thank Ravi Bansal, John Coleman, Chandra Das, Ronald Gallant, Uday Rajan, Bryan Routledge, Thomas Tallarini, George Tauchen, Chris Telmer, seminar participants at Carnegie Mellon University and Duke University, René Stulz, the editor, and an anonymous referee for helpful comments and suggestions. The article was presented at the 1996 annual meetings of Society for Economic Dynamics and Control in Mexico City. All remaining errors are mine.

that incomplete markets with borrowing constraints do not resolve the equity premium puzzle but they lower the mean risk free rate and increase the volatility of pricing kernels. Telmer (1993) finds that in certain cases, an incomplete market economy with borrowing constraints gives rise to a pricing kernel with a standard deviation approximately seven times larger than that of a representative agent model.

The borrowing constraints used in all the studies cited above take the form of a lower bound on an investor's bond holdings, which is a certain percentage of total income or per capita income. For example, Lucas (1994) chooses the borrowing constraint by allowing an investor to borrow up to 20 percent of total income (or 40 percent of per capita income in an economy with two types of agents). Telmer (1993) uses borrowing constraints which restrict an investor's bond holdings to be either greater than or equal to -25 percent of total income or no less than negative the minimum of the investor's one-time exogenous endowment income.[1] In the absence of cross-sectional data on consumer credit limits, it is difficult to resolve which one of the borrowing constraints used above is more reasonable. More important, the aforementioned borrowing constraints are independent of individual characteristics and income streams which in reality are important factors in determining an investor's borrowing limit.

As a departure from the above ad hoc borrowing constraints, following the idea of Bewley (undated), Aiyagari (1994) studies an endogenous borrowing constraint that allows an investor to borrow up to the capitalized value of his worst possible sequence of income shocks discounted at some constant interest rate determined by the marginal product of capital. The intuition is to rule out cases in which an investor could hold a portfolio that may admit negative consumption. We thus refer to it as the nonnegative consumption borrowing constraint.

In this article we first generalize the nonnegative consumption borrowing constraint into an economy with stochastic interest rates. We then introduce an alternative endogenous borrowing constraint—the no default borrowing constraint, which allows an investor to borrow no more than what he has incentive to pay back. The goal is achieved by choosing the borrowing limit such that an investor's expected discounted lifetime utility from participating in the asset market is at least as high as that of autarky, in which the investor only consumes his exogenous endowment income every time period. Because the expected discounted lifetime utility from participating in the asset market and autarky are both functions of individual characteristics such as the time preference rate, risk aversion, and income streams; the borrowing constraint thus obtained is endogenous and has the interpretation of being the borrowing limit, which rules out default. We use the term "endogenous" in a broad sense because the above borrowing limit is determined by an outside agency and does

[1] In addition to the ones used in Telmer (1993) and Lucas (1994), Aiyagari and Gertler (1991) choose $\underline{B}$ as being 35 percent of per capita income. Among others, Heaton and Lucas (1996) assume the borrowing constraint to be 10 percent of per capita income.

not depend on an investor's decisions and/or actions (to be discussed in detail later). Thus, the borrowing constraint is exogenous from an investor's point of view.

Using the exogenous driving forces estimated by Heaton and Lucas (1996), based on the annual National Income and Product Account data and the Panel Study of Income Dynamics data, we solve models with different borrowing constraints using an iterative policy function algorithm introduced by Coleman (1990). The quantitative properties of the no default borrowing constraint model are then discussed for an array of structural parameters. The consumption and asset prices are also simulated from the model, and their first and second moments are compared to those found in the observed data for reasonable structural parameters. The Hansen and Jagannathan (1991) methodology is applied to the simulated consumption series to analyze the mean-standard deviation pairs of the stochastic discount factor as a function of individual intertemporal marginal rate of substitution (IMRS) embedded in the model. For comparison, we also calculate the mean-standard deviation loci of stochastic discount factors for a model with an ad hoc borrowing constraint of 35 percent of total income and for a frictionless complete markets model.

The main findings of the article are the following. First, the nonnegative consumption borrowing constraint is found to be very loose, and it does not have much impact on asset returns. For reasonable structural parameter values (a discount factor (β) of 0.98, and a risk aversion coefficient (γ) of 2.0), this borrowing constraint allows an investor to borrow more than seven and a half times the current total income. Imposing the constraint only lowers the mean risk free rate by less than 0.06 percent. Second, the no default borrowing constraints for certain sets of structural parameters can be much less stringent than those used in the current literature, but they are much tighter than the borrowing constraint induced by imposing the nonnegative consumption. Third, the loci of the stochastic discount factors embedded in the models with market frictions lie above the locus of the stochastic discount factor implied by the frictionless complete markets model. Fourth, the no default borrowing constraint model generates a low risk free rate for reasonable structural parameters, but it fails to generate enough volatility observed in the actual bond returns. This implies that other types of market frictions such as transaction costs may be needed in order to resolve the volatility puzzle.

The rest of the article is organized as follows. Section I describes the economic environment and the ad hoc borrowing constraints. Section II discusses the endogenous borrowing constraints. Section III presents the numerical results on the endogenous borrowing constraints and their effect on the pricing kernel and asset returns. Section IV provides concluding remarks.

I. The Economic Environment and Ad Hoc Borrowing Constraints

The economic environment described here is similar to the one used in Telmer (1993), Lucas (1994), and Heaton and Lucas (1995). It features discrete time with uncertainty. Let time be indexed by $t \in \mathbf{T}$, where $\mathbf{T}$ is the set of nonnegative integers. There are two types of infinitely lived investors denoted by $i = 1$, 2, and one perishable consumption good in the economy. Information is common knowledge for all investors. The objective of investor i is to maximize the following expected discounted utility function:

$$E_0\left[\sum_{t=0}^{\infty} \beta^t U^i(c_t^i)\right], \qquad i = 1, 2 \tag{1}$$

where β is the investor's subjective discount factor and c_t^i is his time t consumption. We assume that $U^i(\cdot)$ is continuous, concave, strictly increasing and continuously differentiable and the discount factor β is the same across all investors in the economy.

There is only one asset in the financial market—a pure discount bond. Holding a unit of bond at time t provides a risk free claim to one unit of consumption at time $t + 1$. Investors in the economy receive their income from two sources, endowment and asset payoffs. The first is exogenous and follows a stochastic process. The second is endogenous due to the endogeneity of asset holdings.

ASSUMPTION 1. *Investors cannot trade contracts written on their uncertain future endowment income.*

We make the above assumption because there may exist a moral hazard or adverse selection problem in reality which leads to the missing market although we do not model moral hazard or adverse selection explicitly in our economy. Since a market for contingent claims on future endowment income is not available, the payoffs to the existing asset cannot span the consumption space. Therefore, asset markets are incomplete. It is important to note that we use the concept in a narrow sense. Broadly speaking, market frictions such as transaction costs, short sale constraints, and borrowing constraints also make asset markets incomplete.[2] At time t, investor i's budget constraint is given by:

$$c_t^i + p_t b_t^i \leq y_t^i + b_{t-1}^i \tag{2}$$

where b_t^i and y_t^i are his bond holdings and endowment income at time t, respectively, and p_t is the bond price at time t.

So far, no restrictions have been imposed on the investor's bond holdings. In an economy with complete markets, investors choose optimal portfolio holdings at the initial period and no subsequent trade is necessary (see Huang and Litzenberger (1988), p. 185). Hence, no restriction is needed on individual bond holdings. In the incomplete market economy, however, trading is necessary for investors to effectively smooth their consumption and achieve the maximum expected lifetime utility. Restrictions on portfolio holdings are therefore needed to rule out default and/or Ponzi schemes, and to ensure the existence of

[2] Geanakoplos (1990) gives a thorough survey on incomplete markets.

equilibrium. A common method of achieving this is to impose a priori bounds on investor's portfolio holdings:

$$b_t^i \geq -\underline{B}^i \tag{3}$$

where $\underline{B}^i \geq 0$ and is often chosen to be a certain fraction of either total income or per capita income.

The investor's problem is to maximize his expected discounted lifetime utility subject to a stream of budget constraints and borrowing constraints. Formally, for investor i, we have

$$\max_{\{c_t^i, b_t^i\}_{t=1}^{\infty}} E_0\left[\sum_{t=0}^{\infty} \beta^t U^i(c_t^i)\right] \tag{4}$$

subject to equations (2) and (3).

Definition 1. An equilibrium for the economy is a set of sequences, namely,

$$\{c_t^1, c_t^2, b_t^1, b_t^2, p_t\}_{t=0}^{\infty}$$

such that

1. Each investor in the economy maximizes his expected discounted lifetime utility subject to a stream of budget constraints and borrowing constraints;
2. For each state of the world, commodity and asset markets clear in each period. That is, the optimal consumption and portfolio choice must satisfy the following conditions:

$$c_t^1 + c_t^2 = y_t^1 + y_t^2, \tag{5}$$

$$b_t^1 + b_t^2 = 0. \tag{6}$$

Assuming that the investor's preference is represented by a constant relative risk aversion utility function, we can derive the following first-order conditions that need to be satisfied in equilibrium:

$$p_t c_t^{i-\gamma} = E_t[\beta c_{t+1}^{i-\gamma}] + \mu_t^i \tag{7}$$

$$\mu_t^i(b_t^i + \underline{B}^i) = 0, \qquad \mu_t^i > 0, \qquad \text{if} \quad b_t^i = -\underline{B}^i, \tag{8}$$

and equations (2) and (5), where μ_t^i is the time t Lagrangian multiplier of the borrowing constraint.

Telmer (1993) conjectures that for an incomplete market economy with an ad hoc borrowing constraint, the state space is sufficiently spanned by the exogenous driving forces and the beginning of period bond holdings. Let Z_t be the vector of exogenous variables at time t which consists of the output growth and the individual endowment income. The state space is represented by (Z_t, b_{t-1}).

Many different values have been used for the borrowing constraints in the current literature. The main drawback of such ad hoc borrowing constraints is that they are independent of investor characteristics and income streams, which is at odds with the practice in reality. In the next section, we discuss ways to specify borrowing constraints so that they depend on the economic environment and have a more natural economic interpretation.

II. The Endogenous Borrowing Constraints

A. The Nonnegative Consumption Borrowing Constraint

One way to endogenize the borrowing constraints is to use the present value of an investor's worst future income stream as the upper limit of borrowing. This method is first studied by Bewley (undated) and more recently by Aiyagari (1994). The result is based on the fact that, if $\lim_{c \to 0} u'(c) = \infty$, then the agent will never hold a portfolio which admits some positive probability of nonpositive consumption. Thus, a borrowing constraint is necessarily implied by nonnegative consumption (Aiyagari (1994), p. 666). In Aiyagari's economy, the interest rate is constant and determined by the marginal product of capital. The nonnegative consumption borrowing constraint is thus given by $y_{\min}/r$, where $y_{\min}$ is the minimum income and r is the interest rate.

In our economy, however, the interest rate is stochastic and depends on the realization of exogenous state variables and the beginning of period asset allocation. A natural generalization of the above borrowing constraint is to allow an investor to borrow up to the present value of his worst possible income stream, discounted at the interest rate associated with this income stream, conditional on the constraint binding. If there is more than one exogenous state for which this income realization occurs, then take the state with the highest interest rate.[3] The intuition is as follows. Imagine that an investor has to borrow the maximum amount allowed every period to make ends meet. In the worst case scenario, in which he gets the worst income realization every period, the investor's present value of income is the sum of his worst income stream discounted at the maximum interest rate corresponding to the worst income and the maximum borrowing. If the investor is allowed to borrow more than the above amount, then there is a positive probability that the present value of the worst income stream is less than his debt and the nonnegative consumption condition will be violated.

In a stationary economy in which the individual income has a lower bound $y^i_{\min}$, the nonnegative consumption borrowing constraint can be defined as a fixed point to the following equation:

$$\underline{B}^i = \frac{y^i_{\min}}{\max_{\Omega | y^i = y^i_{\min}} r(y^i_{\min}, -\underline{B}^i)} \tag{9}$$

[3] We thank the referee for suggesting this idea and the numerical algorithm to implement it.

where $\max_{\Omega|y^i=y^i_{\min}} r(y^i_{\min}, -\underline{B}^i)$ represents the maximum interest rate associated with the worst income realization stream conditional on the constraint binding, and $\Omega|y^i = y^i_{\min}$ is the space of outcomes of exogenous variables conditional on income being the minimum. In a growth economy in which the level of individual income has no lower bound, we can define the nonnegative consumption borrowing constraint in relative terms or as a percentage of total income. Let $\tilde{y}^i_{\min}$ be the minimum individual income relative to total income and let g be the growth rate of total income. Then the nonnegative consumption borrowing constraint as a percentage of current total income is a fixed point to the following equation (the derivation is given in Appendix A):

$$\underline{B}^i = \frac{\tilde{y}^i_{\min}}{\max_{\Omega|\tilde{y}^i=\tilde{y}^i_{\min}}\left(\dfrac{r(\tilde{y}^i_{\min}, -\underline{B}^i) - g}{1+g}\right)}. \tag{10}$$

where $\Omega|\tilde{y}^i = \tilde{y}^i_{\min}$ is the space of outcomes of exogenous variables conditional on the relative income being the minimum.

To find the solution to the above borrowing constraint, we use the following algorithm. Start with an arbitrary borrowing limit. Solve for an equilibrium interest rate function.[4] The right-hand side of equation (10) gives us a new borrowing limit. Use the new borrowing limit to solve for next round equilibrium interest rate function. Iterate the above steps until the borrowing limit converges.

B. The No Default Borrowing Constraint

We now introduce an alternative way to endogenize the borrowing constraints. The idea is to find a borrowing constraint which, at each date-state, will guarantee that the investor prefers participating in the asset market to defaulting. Specifically, we assume that there exists an outside agency that knows the investor's problem. The agency plays no role other than in setting up and enforcing the borrowing limits. Should an investor default on his debt, the agency would exclude him from intertemporal asset trading forever. As a result, he would be deprived of the risk sharing opportunities in the future. With this in mind, the agency imposes a borrowing constraint for each investor such that the investor's expected lifetime utility from participating in the asset market is at least as high as that of staying in autarky, where he consumes his exogenous endowment income.

Let $W^i_t = E_t[\Sigma^{\infty}_{j=0} \beta^j U^i(c^i_{t+j})]$ where c^i_{t+j} is consumption along some optimal equilibrium path, and $V^i_t = E_t[\Sigma^{\infty}_{j=0} \beta^j U^i(y^i_{t+j})]$. Then W^i_t and V^i_t are investor i's expected lifetime utilities of participating in the asset market and staying in

[4] The equilibrium interest rate function is solved numerically using Coleman's policy function iteration algorithm. See Appendix C for detail.

autarky, respectively. The borrowing limit is then defined as follows (a recursive definition is introduced subsequently):

$$\underline{B}^i = \min\{-\underline{b}_t^i\colon W_t^i = V_t^i,\ t = 1, 2, \ldots, \infty\} \tag{11}$$

where $\underline{b}_t^i$ is the amount of borrowing such that W_t^i is equal to V_t^i. The borrowing constraint thus defined depends not only on the investor's characteristics such as time preference rate and risk aversion but also on his exogenous endowment income stream $\{y_t^i\}$. Because the constraint can be interpreted as the borrowing limit such that an investor will not default and live in autarky, we refer to it as the "no default borrowing constraint."

Since investors face borrowing constraints set up by the agency, from their view-point the no default borrowing constraints affect their decisions in the same way as the ad hoc borrowing constraints. This makes it possible for us to use the same set of state variables for this economy as for the economy with the ad hoc borrowing constraints. Given the state space, the endogenous variables such as consumption growth, bond holdings, bond prices, and expected discounted utilities are functions of state variables (Z_t, b_{t-1}). We can therefore define the no default borrowing constraint recursively as follows:

$$b_t^i \geq -\min_{Z_t \in \Omega_Z}\{-\underline{b}^i(Z_t)\colon W^i(Z_t, \underline{b}^i(Z_t)) = V^i(Z_t)\} \tag{12}$$

where Ω_Z is the space of outcomes for Z.

The following result shows the existence of the no default borrowing constraint defined above under certain conditions.

PROPOSITION 1. *Suppose that an equilibrium exists for the economy and the equilibrium asset holdings are nontrivial. If the vector of exogenous variables Z_t takes only a finite number of outcomes, and*

$$\lim_{b_{t-1} \to -\infty} W_t^i(Z_t, b_{t-1}) = -\infty, \tag{13}$$

then there exists a no default borrowing limit given by

$$\underline{B}^i = \min_{Z_t \in \Omega_Z}\{-\underline{b}^i(Z_t)\colon W^i(Z_t, \underline{b}^i(Z_t)) = V^i(Z_t)\}. \tag{14}$$

Proof. see Appendix B.

Because of the impossibility of getting a closed-form solution for the model presented above, we resort to numerically approximating the solution. Several numerical methods have been proposed to solve dynamic equilibrium models in general (see Taylor and Uhlig (1990)). We choose the policy function iteration (Coleman (1990)) method to solve our problem. The detailed algorithm is given in Appendix C. The numerical results and discussions are presented in the next section.

III. Numerical Simulation

A. The Exogenous Driving Processes

Because we are dealing with a growth economy, to obtain stationarity, we normalize the individual endowment income, consumption, and bond holdings by the total income, and normalize the Lagrange multiplier by $Y_t^{-\gamma}$ to create $\tilde{y}_t^i$, $\tilde{c}_t^i$, $\tilde{b}_t^i$, and $\tilde{\mu}_t^i$ such that $\tilde{y}_t^i = y_t^i/Y_t$, $\tilde{c}_t^i = c_t^i/Y_t$, $\tilde{b}_t^i = b_t^i/Y_t$, and $\tilde{\mu}_t^i = \mu_t^i/Y_t^{-\gamma}$.

For the exogenous driving processes, we use the results from Heaton and Lucas (1996). Let $Z_t = [\log(Y_t/Y_{t-1}), \log(D_t/Y_t), \log(y_t^1/(Y_t - D_t))]$, where Y_t and D_t are the total output and dividends at time t, respectively, and y_t^1 is investor one's endowment income. They assume the following vector autoregression (VAR) specification for the exogenous process:

$$\begin{bmatrix} Z_{1t} \\ Z_{2t} \\ Z_{3t} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix} \begin{bmatrix} Z_{1t-1} \\ Z_{2t-1} \\ Z_{3t-1} \end{bmatrix} + \begin{bmatrix} E\begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} \\ \epsilon_{3t} \end{bmatrix} \tag{15}$$

where E is a lower triangular matrix.

Heaton and Lucas estimate these parameters using annual aggregate income and dividend data from the National Income and Product Account (NIPA) and annual household income data from the Panel Study of Income Dynamics (PSID). Since the dividends account for only about 3.8 percent of the gross national product (GNP), it is reasonable to use their estimates in our bond economy. We apply the Hermite-Gauss quadrature rule to discretize the above VAR. With each of the two state variables (growth rate of total output and share of investor one's endowment income) taking two outcomes, there are four possible states for the discretized exogenous variables. The high growth rate of output is 4.8 percent and the low is -0.68 percent. The mean of the output growth rate is 2.1 percent with an annual standard deviation of 2.8 percent. The distribution of income between the two investors has a mean of 50 percent and a standard deviation of 12 percent per year.

B. The Nonnegative Consumption Borrowing Constraint

The nonnegative consumption borrowing constraint is a solution to the fixed point problem defined by equation (10). Using the above exogenous processes, we solve the model with the nonnegative consumption borrowing constraint numerically for some reasonable structural parameters. By setting the discount factor (β) at 0.98 and the risk aversion coefficient (γ) at 2.0, we find that the nonnegative consumption borrowing constraint in terms of total current income is 7.53508. In other words, the nonnegative consumption borrowing constraint allows an investor to borrow up to more than seven and one half times the total current income. This is a very loose borrowing constraint and does not have a significant impact on asset returns. This is confirmed by comparing the mean and standard deviation of asset returns in this economy with the mean and standard deviation of asset returns in a complete markets

Table I

Bond Returns with the Nonnegative Consumption Borrowing Constraint

This table shows the mean and standard deviation of bond returns in an economy with the nonnegative consumption borrowing constraint (column denoted "NC"), and in a complete market economy (column denoted "CM"), respectively, for a discount factor (β) of 0.98 and a risk aversion coefficient (γ) of 2.0. Bond returns are obtained by simulating a long series using the equilibrium bond price functions. For the economy with the nonnegative consumption borrowing constraint, the equilibrium bond price function is solved numerically using a policy function iteration algorithm with the nonnegative consumption borrowing constraint imposed. The bond price function for the complete market economy is obtained by evaluating the conditional expectation of the intertemporal marginal rate of substitution (IMRS) using the growth rate of the aggregate consumption.

Moment	NC	CM
Mean	0.06123	0.06177
Std. Dev.	0.00884	0.00550

economy with the same exogenous processes and structural parameters. The result is provided in Table I.

The model with the nonnegative consumption borrowing constraint lowers the mean risk free rate from 0.06177 to 0.06123 and increases the standard deviation from 0.0055 to 0.0088.

C. The No Default Borrowing Constraint

The no default borrowing constraint is determined endogenously in this economy. It is a function of the investor's structural parameters such as the time preference rate, the risk aversion coefficient, and the exogenous driving processes. Specifically, the no default borrowing limit is determined by the following formula:

$$\underline{B} = \min_{k \in K}\{-\tilde{\underline{b}}(k) \colon \tilde{W}(k, \tilde{\underline{b}}(k) = \tilde{V}(k) \tag{16}$$

where K is the number of exogenous states (we omit the superscript i because the two investors are symmetric and their borrowing limits are the same), $\tilde{W}(\cdot, \cdot)$ and $\tilde{V}(\cdot)$ are the transformed and normalized value function, and autarky utility, respectively (see Appendix D for detail). For different sets of structural parameters, we compute the equilibrium and find the borrowing limits. The results are presented in Table II.

For the array of structural parameters used in our numerical simulation (β ranging from 0.95 to 0.98 and γ ranging from 1.05 to 6.0)[5] the no default

[5] We find that the numerical results become sensitive when the discount factor takes higher values than 0.98. Since the model period is taken to be one year, the reasonable range for the discount factor should be between 0.95 and 0.98. See Aiyagari (1994), Heaton and Lucas (1995), Labadie (1989), and Lucas (1994).

Table II

The No Default Borrowing Limits

This table shows the no default borrowing limits ($\underline{B}$) for various combinations of the discount factor (β) and the risk aversion coefficient (γ). For each combination of the discount factor and the risk aversion coefficient, the no default borrowing limit is obtained by numerically solving for the equilibrium using the policy function iteration algorithm with the no default borrowing constraint imposed, i.e., $\tilde{b} \geq \underline{B}$, and $\underline{B} = \min_{k \in K}\{-\underline{\tilde{b}}(k)\colon \tilde{W}(k, \underline{\tilde{b}}(k)) = \tilde{V}(k)\}$, where $\tilde{b}$ is an investor's bond holdings, and $\tilde{W}$ and $\tilde{V}$ are the value function, and the autarky utility, respectively.

	$\underline{B}$			
γ	$\beta = 0.95$	$\beta = 0.96$	$\beta = 0.97$	$\beta = 0.98$
1.05	0.18439	0.26197	0.39164	0.64680
1.10	0.19346	0.27251	0.40139	0.64900
1.30	0.23185	0.31162	0.43673	0.65868
1.50	0.26624	0.34648	0.46749	0.66927
2.00	0.34132	0.42047	0.53135	0.69721
3.00	0.46000	0.53323	0.62703	0.75146
4.00	0.55093	0.61728	0.69793	0.79817
5.00	0.62220	0.68216	0.75251	0.83665
6.00	0.67913	0.73353	0.79591	0.86820

borrowing limits range from 18.4 percent to 86.8 percent of total income, while the ad hoc borrowing constraints of other studies range only from 0 to 35 percent. Thus, the borrowing constraints imposed by the previous studies are much more stringent than what is required to rule out default. The table also reveals that the no default borrowing limit increases as investors become more risk averse (the risk aversion coefficient increases).

Intuitively, there is a tradeoff between the benefits associated with access to future risk sharing and the benefits associated with defaulting and not paying back the debt.[6] If an investor has a low risk aversion coefficient, he does not care very much about risk sharing in the future. Therefore, not paying back the debt becomes relatively attractive and he will default even when he holds a small amount of debt. On the other hand, a highly risk averse investor cares a lot about risk sharing, and he will not default unless he holds a large amount of debt. The same argument also explains the fact that, for a given risk aversion coefficient, when the time preference parameter β increases, the borrowing limit also increases (because as β increases, investors place higher value on future consumption and risk sharing). The explanation provided above thus implies that the borrowing limit is higher when the time preference parameter takes a larger value.

In Table III we report the frequency with which the no default constraints bind. The results are obtained by simulating a long series of bond holdings using the steady state bond holding functions for both investors and then

[6] We thank the referee for suggesting the intuition.

Table III

The Frequency of Binding Borrowing Constraints

This table shows the frequency with which the no default constraints bind for various combinations of the discount factor (β) and the risk aversion coefficient (γ). The results are obtained by simulating a long series of bond holdings using the equilibrium bond holding functions for both investors and then calculating the frequency with which either investor's bond holdings are equal to his borrowing limit.

γ	$\beta = 0.96$	$\beta = 0.98$
1.05	0.02955	0.00840
1.10	0.03040	0.00860
1.30	0.03685	0.00985
1.50	0.04075	0.01000
2.00	0.03280	0.01045
3.00	0.01865	0.00675
4.00	0.00935	0.00345
5.00	0.00410	0.00135
6.00	0.00130	0.00040

calculating the frequency with which either investor's bond holdings are equal to his borrowing limit.[7]

For various parameter combinations the frequency of binding constraints ranges from 0.04 percent to about 4.075 percent. The table reveals that for both discount factors, β of 0.96 and 0.98, the frequency reaches the peak value when the risk aversion coefficient (γ) takes values in the range of 1.5 to 2.0 and then decreases as the risk aversion increases further. This can be attributed to the combined effects of risk aversion and the tightness of the borrowing limit. Initially, as the risk aversion increases, investors borrow more to buffer income shocks and the frequency of facing a binding constraint thus increases. However, as the risk aversion increases further, the borrowing limit becomes loose enough such that investors are less likely to face a binding constraint and the frequency thus decreases.

We also calculate the mean and standard deviation of asset returns embedded in the model for different sets of structural parameters. The results are summarized in Table IV.

For a given discount factor, a higher risk aversion coefficient is associated with a higher average asset return. Intuitively, an investor demands a higher rate of return for a given amount of asset holdings when he is more risk averse. The mean rate of return can be lowered to about 3.6 percent per year, but the corresponding standard deviation is quite low. This implies that the model may not be able to generate risk free rates that can match both the first and second moments of their observed counterparts. For instance, to generate asset returns that match the second moment, the mean rate of return has to be above 5 percent.

[7] We discard the initial 10,000 simulations before we collect the observations (20,000) to remove potential transient effect.

Table IV

Statistics of Bond Returns

This table shows the mean and standard deviation of bond returns implied by the model with the no default borrowing constraint for various combinations of the discount factor (β) and the risk aversion coefficient (γ). The bond returns are obtained by simulating a long series using the equilibrium bond price function, which is solved numerically using the policy function iteration algorithm with the no default borrowing constraint imposed.

	Mean		Std. Deviation	
γ	$\beta = 0.96$	$\beta = 0.98$	$\beta = 0.96$	$\beta = 0.98$
1.05	0.04449	0.03627	0.023184	0.012950
1.10	0.04535	0.03699	0.023785	0.013487
1.30	0.04861	0.03988	0.025910	0.015553
1.50	0.05175	0.04274	0.027935	0.017504
2.00	0.05924	0.04976	0.032297	0.021936
3.00	0.07295	0.06309	0.039107	0.029260
4.00	0.08489	0.07496	0.044122	0.034966
5.00	0.09482	0.08503	0.047728	0.039486
6.00	0.10249	0.09307	0.050249	0.042871

Analyzing the pricing kernel can therefore shed light on why the models fail to match the first two moments. In a Consumption-Based Capital Asset Pricing Model (C-CAPM), the equilibrium bond price with a unit payoff can be represented by:

$$p_t = E_t[m_{t+1}] \tag{17}$$

where m_{t+1} is the pricing kernel or stochastic discount factor, which is determined by the parametric specifications of investor's preferences, market structure, and information flows. In frictionless complete markets, the representative agent's intertemporal marginal rate of substitution (IMRS) is the valid stochastic discount factor because $p_t = E_t[\text{IMRS}_{t+1}]$. If the representative agent's preference can be characterized by a constant relative risk aversion (CRRA) utility function, then his IMRS is given by:

$$\text{IMRS}_{t+1} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \tag{18}$$

where c_{t+1} and c_t are the representative agent's consumption at time $t + 1$ and t, respectively.

In the incomplete market environment, the aggregation theorem does not hold and the representative agent framework breaks down (Luttmer (1995)). To find a valid stochastic discount factor for our model, let us examine the bond price as a function of individual IMRS. The Euler equations give us:

$$p_t = E_t\left[\beta\left(\frac{c^i_{t+1}}{c^i_t}\right)^{-\gamma}\right]i + \frac{\mu^i_t}{c^{i-\gamma}_t}, \qquad i = 1, 2. \tag{19}$$

If $b_t^i > -\underline{B}^i$ for both $i = 1, 2$, then $\mu_t^i = 0$ for $i = 1, 2$, and the above Euler equations reduce to

$$p_t = E_t\left[\beta\left(\frac{c_{t+1}^i}{c_t^i}\right)^{-\gamma}\right], \qquad i = 1, 2. \tag{20}$$

If $b_t^1 = -\underline{B}^1$, then the bond price is determined by the IMRS of the second investor whose borrowing constraint is not binding:

$$p_t = E_t\left[\beta\left(\frac{c_{t+1}^2}{c_t^2}\right)^{-\gamma}\right]. \tag{21}$$

In exactly the same manner, we get the bond price when investor 2 faces the binding borrowing constraint:

$$p_t = E_t\left[\beta\left(\frac{c_{t+1}^1}{c_t^1}\right)^{-\gamma}\right]. \tag{22}$$

The above analysis allows us to choose the following stochastic discount factor for our incomplete market economy with the borrowing constraint:

$$m_{t+1} = \begin{cases} \beta\left(\dfrac{c_{t+1}^1}{c_t^1}\right)^{-\gamma} & b_t^2 = -\underline{B}^2 \\ \beta\left(\dfrac{c_{t+1}^1}{c_t^1}\right)^{-\gamma} \quad \text{or} \quad \beta\left(\dfrac{c_{t+1}^2}{c_t^2}\right)^{-\gamma} & b_t^i > -\underline{B}^i, \qquad i = 1, 2 \\ \beta\left(\dfrac{c_{t+1}^2}{c_t^2}\right)^{-\gamma} & b_t^1 = -\underline{B}^1. \end{cases} \tag{23}$$

It can be verified that the above stochastic discount factor satisfies $p_t = E_t[m_{t+1}]$ for all scenarios that we have discussed above. This stochastic discount factor has been introduced and analyzed in Telmer (1993) for different levels of investor heterogeneity.

Given the above specification of the pricing kernel, we can simulate a consumption growth series then calculate the mean and standard deviation pairs of the stochastic discount factor for our model for different risk aversion coefficients, ranging from 1.05 to 6.0, while holding the discount factor constant at 0.98. For comparison, we also calculate the mean and standard deviation pairs of the stochastic discount factors implied by the model with an ad hoc 35 percent borrowing constraint and the frictionless complete market model. The results are plotted in Figure 1.

The loci of mean-standard deviation pairs of the stochastic discount factors for all three models are downward-sloping, with the locus for the model with a 35 percent borrowing constraint lying on the top and the locus for the frictionless complete market model on the bottom. This result indicates that for a given mean value of the stochastic discount factor, the model with a 35 percent

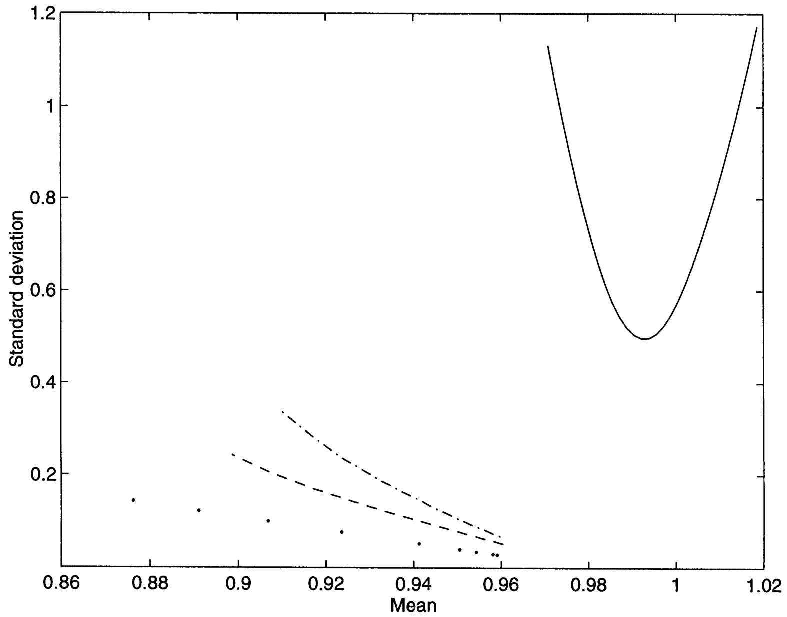

Figure 1. Mean and Standard Deviation Frontier of Stochastic Discount Factors. The solid line is the Hansen-Jagannathan volatility bound constructed using the annual stock and bond returns by compounding quarterly real value-weighted NYSE and three month Treasury bill returns from 1947:Q1 to 1993:Q4 (see Cochrane and Hansen (1992)). The dotted line is the mean and standard deviation locus for a complete market economy. The dashed line is the mean and standard deviation locus for a no-default borrowing constraint. The broken line is the mean and standard deviation locus for a 35 percent ad hoc borrowing constraint. The discount factor (β) is fixed at 0.98.

borrowing constraint generates the highest volatility of the stochastic discount factor and the model with complete markets generates the lowest volatility. Considering that the no default borrowing limits are much less stringent than 35 percent (the borrowing limits vary in the range of 64.7 percent to 86.8 percent as the risk aversion coefficient increases from 1.05 to 6.0), our results are consistent with the findings in Telmer (1993).

Although the loci for models with market frictions lie above the locus for the complete markets model, all three loci fall below the volatility bound on the stochastic discount factor proposed by Hansen and Jagannathan.[8] This would explain why our model is unable to match both the first and second moments.

D. Asset Returns: A Comparison

In Table V we present a comparison of the means and standard deviations of bond returns for: a) the model with complete markets (column denoted "CM"),

[8] The annual Hansen-Jagannathan volatility bound is constructed using the annual stock and bond returns by compounding quarterly real value-weighted NYSE and Treasury bill returns from 1947:Q1 to 1993:Q4 (see Cochrane and Hansen (1992)). We thank Amir Yaron for providing us the MATLAB code used in the computation.

Table V

Bond Returns: A Comparison

This table shows the mean and standard deviation of bond returns implied by the data (column denoted "Data"—taken from Heaton and Lucas (1996)), the model with complete matches (column denoted "CM"), the model with the no default borrowing constraint (column denoted "ND"), the model with a 35 percent fixed borrowing constraint (column denoted "FB"), and the model with a representative agent, using one of the agent's endowment process as if it were the aggregate endowment (column denoted "IN"). The discount factor (β) takes a value of 0.98 and the risk aversion coefficient (γ) is set at 1.5.

Moment	Data	CM	ND	FB	IN
Mean	0.008	0.052	0.043	0.032	0.017
Std. Dev.	0.026	0.004	0.018	0.027	0.197

b) the model with a no default borrowing constraint (column denoted "ND"), c) the model with an ad hoc borrowing constraint of 35 percent of total income (column denoted "FB"), and d) the model with a representative agent, using one of the agent's endowment process as if it were the aggregate endowment (column denoted "IN"). This is intended to assess whether the presence of nontradable endowment has the potential to explain the observed low risk free rate (see Heaton and Lucas, 1996).

The results indicate that model (d) performs the best in terms of matching the first and second moments of the observed bond returns, followed by the ad hoc borrowing constraint model. The model with no default borrowing constraint and the complete market model finish second to last and last, respectively. The performance of these models is clearly dependent on the extent to which investors in the economy can diversify their idiosyncratic risks. Allowing investors even limited access to the bond market helps them to effectively diversify their idiosyncratic risks. As a result, individual consumption growth is less volatile and the bond returns are less variable. For instance, the model with the ad hoc constraint performs better than the no default model because the 35 percent borrowing limit is a tighter bound than the no default borrowing limit which is 67 percent for β equal to 0.98 and γ equal to 1.5. Both models perform better than the CM model. However, except for model (d), none of the three models in which investors have access to asset markets can match both the first and second moments of bond returns with their observed counterparts for the given combination of structure parameters. This is consistent with the findings of previous studies (Telmer (1993) and Lucas (1994) among others) and can be traced back to the behavior of the stochastic discount factors discussed above.

IV. Summary

In this article we endogenize the borrowing constraints used in a class of computable incomplete market economies. We first generalize Aiyagari's (1994) nonnegative consumption borrowing constraint into an economy with stochastic interest rates. We find that this constraint is very loose and does not have much impact on asset returns. We then introduce an alternative approach to endogenize the borrowing constraints. The alternative constraint has the interpretation of being the borrowing limit such that investors will not default. It is a function of structural parameters and the exogenous driving processes. Using an iterative policy function algorithm, we find the borrowing limits for an array of structural parameters and a given exogenous stochastic driving process. For the array of structural parameters used in the study, the no default borrowing limits can be much less stringent than the ad hoc borrowing constraints used in the current literature, but they are much tighter than the nonnegative consumption borrowing limit.

The incomplete market model with the no default borrowing constraint fails to match the first and second moments of bond returns with their observed values. This can be explained by the poor performance of the mean and standard deviation locus of the pricing kernel in terms of the Hansen and Jagannathan volatility bound.

An interesting extension of this study would be to find how individual portfolio holdings are affected by restricting the investor's expected discounted utility from participating in the asset market to be at least as high as autarky utility when the financial market consists of a rich array of assets. For instance, if both bonds and stocks are traded, then there may exist many no default portfolios consisting of both assets but not one on any individual asset alone. An investor may be able to short any amount of one asset if he holds enough positive amount of the other asset. This may potentially weaken the effects of wealth constraints on rich individuals and strengthen the effects of the constraints on poor individuals.

Appendix

A. Derivation of the Relative Present Value Borrowing Constraint

Let $\{y^i_\tau\}^\infty_{\tau=t}$ be investor i's income stream. Let $\{r_\tau\}^\infty_{\tau=t}$ be the corresponding interest rate sequence. The present value of the income stream at time t is given by

$$\frac{y^i_{t+1}}{1+r_{t+1}} + \frac{y^i_{t+2}}{(1+r_{t+1})(1+r_{t+2})} + \cdots + \frac{y^i_{t+\tau}}{\Pi^\tau_{j=1}(1+r_{t+j})} + \cdots \tag{A1}$$

Dividing the above summation by the total current income Y_t yields

$$\frac{\tilde{y}^i_{t+1}}{\dfrac{1+r_{t+1}}{1+g_{t+1}}} + \frac{\tilde{y}^i_{t+2}}{\left(\dfrac{1+r_{t+1}}{1+g_{t+1}}\right)\left(\dfrac{1+r_{t+2}}{1+g_{t+2}}\right)} + \cdots + \frac{\tilde{y}^i_{t+\tau}}{\displaystyle\prod_{j=1}^{\tau}\left(\dfrac{1+r_{t+j}}{1+g_{t+j}}\right)} + \cdots, \tag{A2}$$

where $\tilde{y}^i_{t+j}$ is investor i's income at time $t+j$ as a percentage of total income at time $t+j$, and g_{t+j} is the growth rate of total income at time $t+j$.

Let $\tilde{y}^i_{\min}$ be the investor's worst income realization and let $\underline{B}^i$ be the borrowing limit in terms of total income. The present value borrowing constraint is thus a fixed point to the following equation:

$$\underline{B}^i = \frac{\tilde{y}^i_{\min}}{\max_{\Omega|\tilde{y}^i=\tilde{y}^i_{\min}} \frac{1+r(\tilde{y}_{\min}, -\underline{B}^i)}{1+g}} + \cdots + \frac{\tilde{y}^i_{\min}}{\max_{\Omega|\tilde{y}^i=\tilde{y}^i_{\min}} \left(\frac{1+r(\tilde{y}_{\min}, -\underline{B}^i)}{1+g}\right)^{\tau}}$$

$$+ \cdots = \frac{\tilde{y}_{\min}}{\max_{\Omega|\tilde{y}=\tilde{y}_{\min}} \left(\frac{r(\tilde{y}_{\min}, -\underline{B}^i)-g}{1+g}\right)}. \tag{A3}$$

B. Proof of Proposition 1

The investor i's problem at time t is to solve the following Lagrange function:

$$W^i(Z_t, b^i_{t-1}) = \max_{c^i} E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} [U(c^i_s) + \mu^i_s (b^i_s + \underline{B}^i)] \right\}, \tag{B1}$$

$$c^i_s = y^i_s + b^i_{s-1} - p_s b^i_s, \qquad \forall s \geq t, \tag{B2}$$

$$\mu^i_s \geq 0 \quad \text{and} \quad \mu^i_s = 0 \qquad \text{iff} \quad b^i_s + \underline{B}^i > 0, \quad i = 1, 2, \tag{B3}$$

where μ^i_t is the Lagrange multiplier of the borrowing constraint at time t. Denote investor i's wealth at time t as x^i_t, then $x^i_t = y^i_t + b^i_{t-1}$. Differentiating the indirect utility function with respect to wealth and using the Envelope Theorem, we get:

$$\frac{\partial W^i_t}{\partial x^i_t} (Z_t, b^i_{t-1}) = U'(c^i_t), \qquad i = 1, 2. \tag{B4}$$

Since $U'(c^i_t)$ is strictly positive, we have

$$\frac{\partial W^i_t}{\partial x^i_t} (Z_t, b^i_{t-1}) > 0, \qquad i = 1, 2. \tag{B5}$$

Because bond holdings are nontrivial in equilibrium, there exists a $\hat{b}^i_{t-1}$ such that $W^i(Z_t, \hat{b}^i_{t-1}) \geq V^i(Z_t)$. Since Z_t takes finite outcomes, $V^i(Z_t)$ is also finite under assumptions 1 and 2. Because

$$\lim_{b_{t-1} \to -\infty} W^i_t(Z_t, b_{t-1}) = -\infty, \tag{B6}$$

and $(\partial W^i_t / \partial x^i_t)(Z_t, b^i_{t-1}) > 0$, there must exist a $\tilde{b}^i_{t-1}$ such that $W^i(Z_t, \tilde{b}^i_{t-1}) \leq V^i(Z_t)$. According to the intermediate value theorem, there exists a $\underline{b}^i_{t-1}$, where $\tilde{b}^i_{t-1} \leq \underline{b}^i_{t-1} \leq \hat{b}^i_{t-1}$ such that $W^i(Z_t, \underline{b}^i_{t-1}) = V^i(Z_t)$ and $W^i(Z_t, b^i_{t-1}) \geq V^i(Z_t)$ if and only if $b^i_{t-1} \geq \underline{b}^i_{t-1}$.

Given that Z_t takes a finite number of outcomes, denote $\underline{B}^i = \min_{Z_t \in \Omega_Z}\{-\underline{b}^i(Z_t)\}$ where Ω_Z is the space of outcomes for Z, then for any b^i such that $W^i(Z_t, b^i) \geq V^i(Z_t)$ for all Z_t, we have

$$b^i \geq -\underline{B}^i.$$

The above condition implies that $-\underline{B}^i$ is the lower bound on b^i.

To prove that $\underline{B}^i$ is the borrowing limit, we only need to show that $\underline{B}^i$ is positive. We first show that $\underline{B}^i$ is nonnegative. At any time t, if $b_{t-1} = 0$, then $c_\tau = y_\tau$ and $b_\tau = 0$, $\forall \tau \geq t$ satisfy both the budget constraint and $W_t^i(Z_t, 0) \geq V_t^i(Z_t)$. Since $W_t^i(Z_t, \underline{b}_t^i) = V_t^i(Z_t)$ and $W_t^i(Z_t, .)$ is an increasing function, it must be true that $\underline{b}_t^i \leq 0$. According to the definition of $\underline{B}^i$, it is nonnegative. Since the equilibrium bond holdings are nontrivial, $\underline{B}^i$ can not be 0. Therefore it is positive and the no default borrowing limit. Q.E.D.

C. Algorithm

The dynamic model presented in Sections II and III gives rise to the following set of equations that has to be satisfied in equilibrium.

$$p_t[\tilde{c}_t^i]^{-\gamma} = E_t\beta[\tilde{c}_{t+1}^i]^{-\gamma} + \tilde{\mu}_t^i \tag{C1}$$

$$\tilde{c}_t^i + p_t\tilde{b}_t^i \leq \tilde{y}_t^i + \tilde{b}_{t-1}^i/g_t \tag{C2}$$

$$\tilde{c}_t^1 + \tilde{c}_t^2 = \tilde{y}_t^1 + \tilde{y}_t^2 \tag{C3}$$

$$\tilde{b}_t^1 + \tilde{b}_t^2 = 0 \tag{C4}$$

$$\tilde{\mu}_t^i(\tilde{b}_t^i + \underline{B}^i) = 0, \quad \tilde{\mu}_t^i > 0, \qquad \text{if} \quad \tilde{b}_t^i + \underline{B}^i = 0 \tag{C5}$$

where the borrowing limit $\underline{B}^i$ is given by either the present value budget balance borrowing constraint

$$\underline{B}^i = \frac{\tilde{y}_{\min}^i}{\max_{\Omega|\tilde{y}^i = \tilde{y}_{\min}^i}\left(\dfrac{r(\tilde{y}_{\min}^i, -\underline{B}^i) - g}{1+g}\right)} \tag{C6}$$

or by the no default borrowing constraint

$$\underline{B}^i = \min_{Z_t \in \Omega_Z}\{-\tilde{\underline{b}}^i(Z_t)\colon \tilde{W}^i(Z_t, \tilde{\underline{b}}^i(Z_t)) = \tilde{V}^i(Z_t)\}. \tag{C7}$$

We assume that the laws of motion for the exogenous variables can be approximated by a finite state Markovian process and that the stationary equilibrium exists. Then the above equations can be cast into the following framework, in which a modified Coleman's policy function iteration algorithm can be used to find decision rules.

$$p(Z, X)[\tilde{c}^i(Z, X)]^{-\gamma} = E[\beta[\tilde{c}^i(Z', X')]^{-\gamma}] + \tilde{\mu}^i(Z, X) \qquad \text{(C8)}$$

$$\tilde{c}^i(Z, X) + p(Z, X)\tilde{b}^i(Z, X) \leq \tilde{y}^i + \tilde{b}^i/g \qquad \text{(C9)}$$

$$\sum_{i=1}^{2} \tilde{c}^i(Z, X) = \sum_{i=1}^{2} \tilde{y}^i \qquad \text{(C10)}$$

$$\sum_{i=1}^{2} \tilde{b}^i(Z, X) = 0 \qquad \text{(C11)}$$

$$\tilde{\mu}^i(Z, X)(\tilde{b}^i(Z, X) + \underline{B}^i) = 0, \quad \tilde{\mu}^i(Z, X) > 0, \qquad \text{if} \quad \tilde{b}^i(Z, X) + \underline{B}^i = 0 \qquad \text{(C12)}$$

The transformed and normalized value function and autarky utility can be expressed as follows

$$W^i(Z, X) = \frac{(\tilde{c}^i(Z, X))^{1-\gamma}}{1-\gamma} + \beta E\tilde{W}^i(Z', X'), \qquad \text{(C13)}$$

$$\tilde{V}^i(Z) = \frac{(\tilde{y}^i)^{1-\gamma}}{1-\gamma} + \beta E\tilde{V}^i(Z'), \qquad \text{(C14)}$$

where $X = \tilde{b}^i$ is the beginning of period bond holdings and $Z = (g, \tilde{y}^i)$ is a vector of exogenous state variables. Z' and X' are the counterparts of Z and X in next period. The expectation is taken over Z'. Denote h as the control function which consists of consumption allocation, asset holdings, asset price and all the Lagrangian multipliers. Let $\mathcal{T}$ be the nonlinear operator such that $\mathcal{T}[h\ \tilde{W}]'$ satisfies the above equations. Then, an equilibrium control and value function is a fixed point $[h\ \tilde{W}]' = \mathcal{T}[h\ \tilde{W}]'$ of the nonlinear operator $\mathcal{T}$. We assume that $\mathcal{T}$ exists and is well defined. Then, the nonlinear operator $\mathcal{T}$ can be obtained by the Gauss-Seidel iterations, i.e.,

$$[h_{n+1}\tilde{W}_{n+1}]' = \mathcal{T}[h_n\tilde{W}_n]', \qquad n \geq 0 \qquad \text{(C15)}$$

converges for a given $[h_0\ \tilde{W}_0]'$.

Specifically, the following steps are involved in approximating a fixed point of $\mathcal{T}$. First, use the Hermite-Gauss quadrature rule to discretize the exogenous driving process to get a finite state Markov chain along with its transition probability matrix. Second, define a grid

$$\mathcal{D} = (Z_k, \tilde{b}_j), \quad j = 1, 2, \ldots, J; \qquad k = 1, 2, \ldots, K. \qquad \text{(C16)}$$

The lower bound on $\tilde{b}$ will be determined when the equilibrium is found. We give certain initial value to set up the grid. Third, using the quadrature rule, we set up a set of linear equations on $\tilde{V}^i$, and solve them to get the values of $\tilde{V}^i(Z)$ on the grid D. See Tauchen and Hussey (1991) for detail. Fourth, define the finite-dimensional set $[H_D\ \tilde{W}_D]'$ in which a typical element is a function $[h\ \tilde{W}]'$ that consists of values on the grid D along with an interpolation rule to compute the values off the grid. Fifth, given an initial function $[h_0\ \tilde{W}_0]' \in [H_D\ \tilde{W}_D]'$, compute $[h_1\ \tilde{W}_1]$ on the grid D such that

$$p(Z, X)[\tilde{c}^i(Z, X)]^{-\gamma} = \beta \sum_{k=1}^{K} [\tilde{c}_0^i(Z', X')]^{-\gamma} \pi(k|m) + \mu^i(Z, X), \tag{C17}$$

$$\tilde{W}^i(Z, X) = \frac{(c^i(Z, X))^{1-\gamma}}{1-\gamma} + \beta \sum_{k=1}^{K} \tilde{W}_0^i(Z', X')\pi(k|m), \tag{C18}$$

and equations (C9), (C10), and (C11) for $m = 1, 2, \ldots, K$, where K is the number of exogenous states. Sixth, use the following formula to find the lower bound on $\tilde{b}^i$ if the present value budget balance borrowing constraint is imposed.

$$\underline{B}^i = \frac{\min_{k\in K}\{\tilde{y}^i(k)\}}{\max_{k\in K_y}\left(\dfrac{r(k, -\underline{B}_0^i) - g}{1+g}\right)} \tag{C19}$$

where K_y is the subset of K such that if $k \in K_y$ then $\tilde{y}^i(k) = \min_{k\in K}\{\tilde{y}^i(k)\}$ and $\underline{B}_0^i$ is the lower bound from the previous round iteration, or alternatively, use the following if the no default borrowing constraint is imposed

$$\underline{B}^i = \min_{k\in K}\{-\underline{\tilde{b}}^i(k)\colon \tilde{W}^i(Z(k), \underline{\tilde{b}}^i(j)) = \tilde{V}^i(Z(k))\} \tag{C20}$$

Use $\underline{B}^i$ as the lower bound for $\tilde{b}^i$ to update the grid D. Seventh, use the $[h_1\ \tilde{W}_1]$ as next $[h_0\ \tilde{W}_0]$ and iterate until $[h\ \tilde{W}]$ converges according to some preset criteria.

D. Transformation and Normalization

Denote

$$\hat{W}_t^i = E_t \sum_{j=0}^{\infty} \beta^j \frac{(c_{t+j}^i)^{1-\gamma}}{1-\gamma} \quad \text{and} \quad \hat{V}_t^i = E_t \sum_{j=0}^{\infty} \beta^j \frac{(y_{t+j}^i)^{1-\gamma}}{1-\gamma}.$$

Since $W_t^i \geq V_t^i$ is equivalent to $\hat{W}_t^i \geq \hat{V}_t^i$, we use $\hat{W}_t^i$ and $\hat{V}_t^i$ to replace W_t^i and V_t^i, respectively. By definition

$$\hat{W}_t^i = \frac{(c_t^i)^{1-\gamma}}{1-\gamma} + E_t \sum_{j=1}^{\infty} \beta^j \frac{(c_{t+j}^i)^{1-\gamma}}{1-\gamma} = \frac{(c_t^i)^{1-\gamma}}{1-\gamma} + \beta E_t \sum_{j=0}^{\infty} \beta^j \frac{(c_{t+1+j}^i)^{1-\gamma}}{1-\gamma}$$
$$= \frac{(c_t^i)^{1-\gamma}}{1-\gamma} + \beta E_t \hat{W}_{t+1}^i. \qquad \text{(D1)}$$

Dividing through by $Y_t^{1-\gamma}$ and denoting $\tilde{W}_t^i = \hat{W}_t^i / Y_t^{1-\gamma}$ gives

$$\tilde{W}_t^i = \frac{1}{1-\gamma}\left(\frac{\hat{c}_t^i}{Y_t}\right)^{1-\gamma} + \beta E_t\left[\frac{\hat{W}_{t+1}^i}{Y_{t+1}^{1-\gamma}} \frac{Y_{t+1}^{1-\gamma}}{Y_t^{1-\gamma}}\right] = \frac{(\tilde{c}_t^i)^{1-\gamma}}{1-\gamma} + \beta E_t\left[\frac{\tilde{W}_{t+1}^i}{(1+g_{t+1})^{\gamma-1}}\right]. \qquad \text{(D2)}$$

Similarly, we can transform $\hat{V}_t^i$ into $\tilde{V}_t^i$ such that

$$\tilde{V}_t^i = \frac{(\tilde{y}_t^i)^{1-\gamma}}{1-\gamma} + \beta E_t\left[\frac{\tilde{V}_{t+1}^i}{(1+g_{t+1})^{\gamma-1}}\right]. \qquad \text{(D3)}$$

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