Macroeconomic Dynamics, **1**, 1997, 228–254. Printed in the United States of America.

ENDOGENOUS SHORT-SALE CONSTRAINT, STOCK PRICES AND OUTPUT CYCLES

HAROLD H. ZHANG
Graduate School of Industrial Administration, Carnegie Mellon University

This study examines the effect of short-sale constraints on a stock market, in particular, on stock prices, trading volume, and the relationship between stock price movements and output cycles. The economic model features incomplete markets and heterogeneous agents. The short-sale constraint is endogenously determined in the economy and is a function of agents' risk aversion, time preference, and exogenous driving forces. The dynamic model is solved using a policy function iteration algorithm. We find that, for an array of reasonable time-preference parameters and risk-aversion coefficients, the short sale limits range from 27 to 45% of total outstanding shares. Imposing short-sale constraints causes stock prices to move upward. Trading volume is high when some agents have a large amount of stock holdings but incur a negative shock on their nonfinancial income and is low when some agents have few stock holdings and also incur a negative shock to their nonfinancial income. Stock prices are found to be countercyclical and the expected stock returns are procyclical. These countercyclical stock-price movements are shown to be related to the imposition of a short-sale constraint.

Keywords: Short-Sale Constraints, Stock Prices, Output Cycles

1. INTRODUCTION

Short selling of financial assets such as stocks is a common practice for hedging and speculating in financial markets. How much investors are allowed to go short, however, has posed a problem for both researchers and practitioners involved in stock markets. Current research on stock markets has taken the following approaches on the issue of short sales. They either assume that investors can engage in unlimited short selling or assume that investors are not allowed to short at all. The first is used primarily in a complete market economy in which optimal stock holdings are formed at the initial period and no subsequent trade is necessary. The second is used in incomplete-market economies and often justified by arguments such as short selling entails high costs and may result in default. Both assumptions are, in fact, compromises due to the difficulty of finding a short-sale constraint without making arbitrary assumptions.

I thank George Pennacchi, Uday Rajan, Kristian Rydqvist, Tony Smith, Fallaw Sowell, Chris Telmer, Cheng Wang, and Stan Zin for helpful discussions and two anonymous referees and the participants at the Conference on Computation and Estimation in Finance and Economics at Washington University in St. Louis for their comments. All remaining errors are mine. Address correspondence to: Harold H. Zhang, Graduate School of Industrial Administration, Carnegie Mellon University, Pittsburgh, PA 15213 USA; e-mail: huibingz@andrew.cmu.edu.

 1365-1005/97 $9.00 + .10

Assuming that traders have exogenous heterogeneous expectations on the future payoffs of stocks, Miller (1977), Jarrow (1980), and Figlewski (1981) find that imposing short-sale constraint affects stock prices. In particular, Miller (1977) and Figlewski (1981) find that short-sale constraints bias stock prices upward because short-sale constraints underweigh traders with unfavorable expectations. Jarrow (1980) argues that they can bias stock prices in either direction depending on the covariance matrix of future payoffs perceived by traders. However, under the assumption of homogeneous beliefs about the covariance matrix of future payoffs, short-sale constraints will only increase risky asset prices. Using 414 stocks included in the S&P 500 Index, Figlewski (1981) finds that stocks with relatively high short interest[1] have significantly lower risk-adjusted returns in the following period than do the stocks with relatively low short interest. Though some researchers argue that the introduction of derivatives such as options attenuates the effect of short-sale constraints [Figlewski and Webb (1993)], there are still many stocks that are not optionable,[2] and the effect of short-sale constraints on these stocks cannot be ignored. Although the research cited above provides useful insights on the effect of imposing a short-sale constraint on stock prices, they are all static mean-variance analysis with some fixed short-sale limit.

This study introduces an endogenous short-sale constraint[3] into a dynamic incomplete-asset-market economy with heterogeneous agents. The constraint has the property that at each time period, an agent's expected lifetime utility from trading assets with other agents is at least as high as what she obtains if she stays in an autarky position in which no asset trading takes place and the agent consumes her exogenous endowment only. It is therefore a function of structural parameters such as agent's risk aversion, time preference, and the law of motion of exogenous driving forces and has the interpretation of the short-sale limit that agents won't violate. We then examine the effect of the constraint on stock prices, trading volume, and the relationship between stock-price movements and output cycles.

Because closed-form solutions are difficult to find for the economies with incomplete markets, we resort to numerical approximations. A policy function iteration algorithm has been found to work well for this type of model. Specifically, our numerical analysis takes the following steps. First, the effect on the endogenous short-sale constraint is analyzed for different structural parameters such as agent's risk-aversion coefficient and subjective discount factor. Second, the effects of short-sale constraints on consumption allocation, stock holdings, and stock prices are examined for economies with different short-sale constraints. This is accomplished by comparing the equilibrium decision rules obtained for various economies. Third, stock returns and trading volume are simulated from various economies, and their sample statistics are obtained and compared to their counterparts from actual data. Fourth, the relationship between the stock-price movements and output cycles is investigated to see if the economic model under study can explain the empirical findings of Fama and French (1989) on business conditions and expected stock returns.

The main findings of the study can be summarized as follows. First, for reasonable risk-aversion coefficients and discount factors, individual stock holdings can fall below zero. This means that short sales occur in the economy. For a fixed risk-aversion coefficient, the short-sale limit increases as the discount factor increases. However, for a fixed discount factor, the effect on the short-sale limit of changing the risk-aversion coefficient is not monotonic. The short-sale limit first increases and then decreases (in absolute values).

Second, for all values of the exogenous state variables, the decision rules as functions of beginning-of-period stock holdings are similar for the two incomplete market economies under study. For these two economies, both individual consumption and stock holdings are increasing functions of beginning-of-period stock holdings. Stock prices, however, are U-shaped curves. When beginning-of-period stock holdings are evenly distributed, stock prices do not change very much. However, when stock holdings are skewed, stock prices can go up sharply as a result of a binding short-sale constraint. This is consistent with the results of Miller (1977), Jarrow (1980), and Figlewski (1981).

Third, for reasonable parameter values, the economic model fails to match the mean and the standard deviation of the observed stock returns. Although the mean stock return can be matched by increasing the risk-aversion coefficient, it is more difficult to match the standard deviation. For instance, holding the discount factor constant at 0.98 and increasing the risk-aversion coefficient to 5.0, the mean stock return increases to 7.63% per year, which accounts for 86% of its observed counterpart. The corresponding standard deviation, however, is only slightly over 50% of its observed counterpart. The summary statistics indicate that the model with no short sales generates the highest return volatility, the model with a complete market generates the least return volatility, and the model with the endogenous short-sale constraint lies in between. The average trading volume is also higher for the model with the endogenous short-sale constraint than that for the model with no short sales.

Fourth, stock price movements are countercyclical. When the economy is in a good state (i.e., the output growth rate is high), stock prices are generally lower than when the economy is in a bad state (i.e., output growth rate is low). The price differences across states become wider as the beginning-of-period stock holdings become more skewed. We also find that the expected stock returns are procyclical. They are low when the output growth rate is low and high when the output growth rate is high. This is robust for both complete and incomplete markets. The finding is nonetheless in contrast to the empirical results of Fama and French (1989), which document that the expected stock returns are lower when the economic conditions are strong and higher when economic conditions are weak; in other words, the expected stock returns are countercyclical. Possible explanations for the failure lie in the economic model under study. We assume that agent's preferences are the standard time-separable power utility function. Some studies [Campbell and Cochrane (1994)] have shown that introducing a time-nonseparable utility function may help in generating the countercyclical expected stock returns.

We also have abstracted from physical investment, which may contribute to the procyclical relationship between the expected stock returns and output growth rates.

The rest of the paper is organized as follows. Section 2 describes the economic model. Section 3 presents some theoretical results. Section 4 discusses the numerical algorithm and the calibration. Section 5 presents numerical simulation results and discussions. Section 6 provides concluding remarks.

2. ECONOMIC MODEL

The model described below is similar to those of Telmer (1993), Lucas (1994), and Heaton and Lucas (1996). It consists of a large number of two types of agents who produce a single perishable consumption good. Each type of agent is endowed with one unit of labor which is inelastically supplied to the production of the consumption good. The output produced by a type i agent in period t, denoted y_{it}, is random and exogenous.

Assumption 1. Agents are not allowed to trade with each other, claims written on the uncertain future output of production of the consumption good.

This assumption is utilized to create an incomplete markets environment. In addition to the output produced by labor input (called endowment from now on), there exists a productive tree that produces the same consumption good. The output of the tree at time t, denoted d_t, is also random and exogenous and distributed among agents according to the shares of claims on the tree owned by each agent. Although agents are not allowed to trade claims on their uncertain future endowment, they can trade shares that they own on the tree. These shares are, in essence, stocks in the economy.

Information is common knowledge for all agents in the economy and its structure can be characterized by an increasing family of σ algebras. The set of exogenous state variables available at time t, $z_t = (g_t, d_t, y_{1t}, y_{2t})$, are components of the information set at the beginning of period t, in which g_t is the growth of total output of the economy.

Assumption 2. $z \in \Omega$, Ω is finite. The law of motion for z follows the Markov process with transition probability matrix $\pi(z' \mid z)$, $y_1 > 0$ and $y_2 > 0$.

An agent's preferences are defined over a sequence of consumption of the single good and are given by the following sum of the expected discounted utility:

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_{it}) \right], \tag{1}$$

where $\beta \in (0, 1)$ is the subjective discount factor, and c_{it} is agent i's consumption at time t.

Assumption 3. $u: \Re_+ \rightarrow \Re$ is a bounded, continuously differentiable, strictly increasing, and strictly concave function, and $u'(0) = \infty$.

At each period, each agent chooses her consumption and shareholdings on the productive tree to maximize her expected discounted utility given above. Her choice of consumption and holding of shares on the tree has to satisfy her budget constraint at period t,

$$c_{it} + p_t a_{it} \leq y_{it} + (p_t + d_t) a_{i(t-1)}, \quad \textbf{(2)}$$

where p_t is the price for acquiring one share of ownership on the tree, $a_{i(t-1)}$ is the shares owned by agent i at the beginning of the period, and a_{it} is her shareholdings at the beginning of next period.

When an economy has complete markets, no restrictions on asset holdings are required because agents consume a constant share of the total output without actually trading assets. In an environment of incomplete markets, certain restrictions have to be put on agents' shareholdings to rule out cases in which agents take extreme short positions on the stock market.

To allow for short sales and at the same time avoid imposing an arbitrary short-sale constraint, we introduce an endogenous short-sale constraint into the model. For agent i, the endogenous short-sale constraint at period t is defined as:

$$a_{it} \geq A_i, \quad \textbf{(3)}$$

$$A_i = \max\left\{ \underset{\sim}{a}_{it} : E_t\left\{ \sum_{j=0}^{\infty} \beta^j u\left[c_{i(t+j)}\right] \right\} = E_t\left\{ \sum_{j=0}^{\infty} \beta^j u\left[y_{i(t+j)}\right] \right\}, \right.$$
$$\left. t = 1, 2, \ldots, \infty \right\}, \quad \textbf{(4)}$$

where $c_{i(t+j)}$ is agent i's consumption at time $t + j$, $y_{i(t+j)}$ is her endowment at time $t + j$, and $\underset{\sim}{a}_{it}$ is agent i's beginning-of-period stockholdings such that the expected utility from participating in the stock market is equal to the expected autarky utility.

The rationale behind this constraint can be illustrated by the following scenario. Suppose there is an economy with such a legal system: If an agent breaches her contract with any other agents in the economy, she would be excluded from intertemporal asset tradings thereafter, and her financial assets would be seized and transferred to her contract holders to compensate for their losses.[4] In such an environment, an agent would never breach if her expected discounted utility from trading assets were at least as high as what she obtains from an autarkic position in which she only consumes her endowment. The endogenous short-sale constraint, which captures the above idea, keeps an agent engaging in asset trading at every period. A similar constraint has been examined by Kehoe and Levine (1993) in a complete-market environment.

3. SOME THEORETICAL RESULTS

The agent's problem is to maximize her expected discounted utility subject to a stream of budget constraints and endogenous short-sale constraints. We define an equilibrium for the economy as follows:

DEFINITION 1. *An equilibrium for the economy is a quintuple of sequences, namely,*

$$\{c_{1t}, c_{2t}, a_{1t}, a_{2t}, p_t\}_{t=0}^{\infty}$$

such that

1. *Each agent in the economy maximizes her expected discounted lifetime utility subject to a stream of budget constraints and endogenous short-sale constraints; and,*
2. *For each state of the world, commodity and asset markets clear at each period, that is, the optimum consumption and portfolio choice must satisfy the following conditions:*

$$c_{1t} + c_{2t} = y_{1t} + y_{2t} + d_t, \tag{5}$$

$$a_{1t} + a_{2t} = 1. \tag{6}$$

In the above definition, we normalized the aggregate stock supply into one unit. In equilibrium, an agent's expected discounted utility is positively related to her beginning-of-period stockholdings; extreme current short positions taken by an agent will lower her expected utility and may result in an expected discounted utility lower than the autarky utility such that the agent wants to switch to autarky. Imposing the endogenous short-sale constraint, therefore, rules out the possibility for the agent to take the extreme short positions on the stock market such that she always has incentive to pay back.

Let $W_{it} = E_t\{\sum_{j=0}^{\infty} \beta^j u[c_{i(t+j)}]\}$ and $V_{it} = E_t\{\sum_{j=0}^{\infty} \beta^j u[y_{i(t+j)}]\}$. The following proposition shows the existence of endogenous short-sale constraint in equilibrium.

PROPOSITION 1. *Suppose that an equilibrium exists for the economy and the equilibrium trading volume is nontrivial. If $W_{it} = W[z_t, a_{i(t-1)}]$ and $W(.,.)$ is of class $\mathcal{C}^{(1)}$ in $a_{i(t-1)}$ for each $z_t \in \Omega$, then there exists a lower bound on $a_{i(t-1)}$ given by*

$$A_i = \max_{z_t \in \Omega}\{\underline{a}_i(z_t) \colon W_{it}[z_t, \underline{a}_i(z_t)] = V_{it}\}, \tag{7}$$

where Ω denotes the set of outcomes of z.

Proof. See Appendix. ■

The essence of the above result is well illustrated graphically. Figure 1 shows the implication of equating the expected utility from participating in the asset market and the expected autarky utility on stockholdings. The horizontal axis is an agent's stockholdings at the beginning of the period. The vertical axis is the

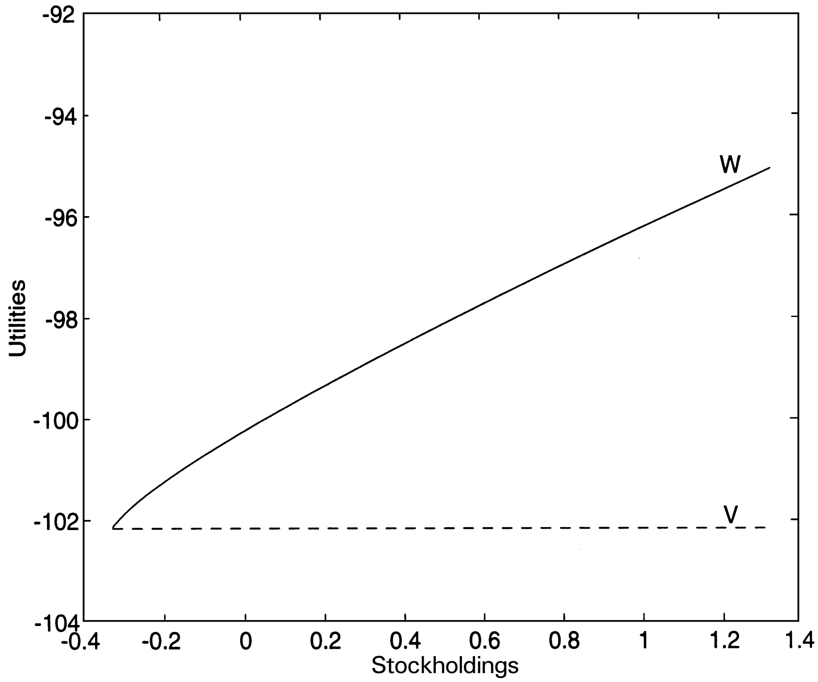

FIGURE 1. Determination of endogenous short-sale limit.

agent's expected discounted utilities from an economy with stock trading (denoted W) and an autarky economy (denoted V), respectively. Because the autarky utility is a function of only exogenous variables, V is a horizontal line in the graph. The expected discounted utility in an economy with stock trading (W), however, is an increasing function of the agent's beginning-of-period stockholdings. W and V intersect at a point. When the agent's beginning-of-period stockholdings are to the right of the horizontal coordinate of the point, W lies above V. When her stockholdings lie to the left of the horizontal coordinate of the point, W falls below V. Let the horizontal coordinate of the intersection be $\underline{a}$. Then, if W is restricted to be at least as high as V, the agent's stockholdings must be greater than or equal to $\underline{a}$.

The lower bound on stockholdings A_i is thus the short sale allowed for agent i in this economy on which she would not default and retreat to an autarky position. Obviously, the short-sale constraint thus determined is a function of structural parameters such as agent's risk-aversion coefficient and subjective utility discount factor. It also depends on the parameters characterizing the underlying driving forces. The same idea also has been used by Zhang (1997) in which an endogenous borrowing constraint is introduced and a bond is the tradable asset in an economy.

Because there is no closed-form solution to the problem under investigation, we resort to numerical approximations. The algorithm is discussed in the next section.

4. NUMERICAL ALGORITHM AND CALIBRATION

4.1. Numerical Algorithm

The economic model presented in Section 2 gives rise to the following set of equations that have to be satisfied in equilibrium.

$$p_t u_c(c_{it}) = \beta E_t\left\{u_c\left[c_{i(t+1)}\right](p_{t+1} + d_{t+1})\right\} + \mu_{it}, \tag{8}$$

$$W_{it} = u(c_{it}) + \beta E_t W_{i(t+1)}, \tag{9}$$

$$V_{it} = u(y_{it}) + \beta E_t V_{i(t+1)}, \tag{10}$$

$$\mu_{it}(a_{it} - A_i) = 0, \quad \mu_{it} > 0, \quad \text{if } a_{it} = A_i, \tag{11}$$

$$A_i = \max\{\underline{a}_{it} : W_{it} = V_{it}, \quad t = 1, 2, \ldots, \infty\}, \tag{12}$$

and equations (2), (5), and (6), where μ_{it} is the time t Lagrangian multiplier of the endogenous short-sale constraint.

By assuming the constant relative risk-aversion-period utility function,

$$u(c_{it}) = \frac{(c_{it})^{1-\gamma} - 1}{1-\gamma}, \tag{13}$$

the model can be calibrated to data after certain normalizations. Specifically, we normalize individual consumption, dividends, and stock prices by the aggregate output. The expected discounted utilities consequently can be normalized as follows. Denote

$$\hat{W}_{it} = E_t \sum_{j=0}^{\infty} \beta^j \frac{\left[c_{i(t+j)}\right]^{1-\gamma}}{1-\gamma} \quad \text{and} \quad \hat{V}_{it} = E_t \sum_{j=0}^{\infty} \beta^j \frac{\left[y_{i(t+j)}\right]^{1-\gamma}}{1-\gamma}.$$

Because $W_{it} \geq V_{it}$ is equivalent to $\hat{W}_{it} \geq \hat{V}_{it}$, we use $\hat{W}_{it}$ and $\hat{V}_{it}$ to replace W_{it} and V_{it}. By definition,

$$\begin{aligned}
\hat{W}_{it} &= \frac{(c_{it})^{1-\gamma}}{1-\gamma} + E_t \sum_{j=1}^{\infty} \beta^j \frac{\left[c_{i(t+j)}\right]^{1-\gamma}}{1-\gamma} \\
&= \frac{(c_{it})^{1-\gamma}}{1-\gamma} + \beta E_t \sum_{j=0}^{\infty} \beta^j \frac{\left[c_{i(t+1+j)}\right]^{1-\gamma}}{1-\gamma} \\
&= \frac{(c_{it})^{1-\gamma}}{1-\gamma} + \beta E_t \hat{W}_{i(t+1)}.
\end{aligned}$$

Dividing through by $Y_t^{1-\gamma}$, where Y_t is the aggregate output at time t, and denoting $\tilde{W}_{it} = \hat{W}_{it}/Y_t^{1-\gamma}$ yields

$$\begin{aligned}\tilde{W}_{it} &= \frac{1}{1-\gamma}\left(\frac{\hat{c}_{it}}{Y_t}\right)^{1-\gamma} + \beta E_t\left[\frac{\hat{W}_{i(t+1)}}{Y_{t+1}^{1-\gamma}}\frac{Y_{t+1}^{1-\gamma}}{Y_t^{1-\gamma}}\right] \\ &= \frac{(\tilde{c}_{it})^{1-\gamma}}{1-\gamma} + \beta E_t \frac{\tilde{W}_{i(t+1)}}{(g_{t+1})^{\gamma-1}}, \end{aligned} \quad (\mathbf{14})$$

where $g_{t+1} = Y_{t+1}/Y_t$ is the gross growth of output. Similarly, we can transform $\hat{V}_{it}$ into $\tilde{V}_{it}$ by dividing through $Y_t^{1-\gamma}$ such that

$$\tilde{V}_{it} = \frac{\left[\tilde{y}_{it}(1-\tilde{d}_t)\right]^{1-\gamma}}{1-\gamma} + \beta E_t \frac{\tilde{V}_{i(t+1)}}{(g_{t+1})^{\gamma-1}}, \quad (\mathbf{15})$$

where $\tilde{y}_{it} = y_{it}/(y_{1t}+y_{2t}) = y_{it}/(Y_t - d_t)$.

We assume that the law of motion for the exogenous variables can be approximated by a finite-state Markov process and that a stationary equilibrium exists. Then the above equations can be rewritten as the following set of equations to which a modified Coleman's (1990) policy function iteration algorithm can be applied to find decision rules.

$$\begin{aligned}\tilde{p}(\tilde{z},a)[\tilde{c}_i(\tilde{z},a)]^{-\gamma} = E\big(\beta\{\tilde{c}_i[\tilde{z}',a'(\tilde{z},a)]\}^{-\gamma}(g')^{-\gamma}\{\tilde{p}[\tilde{z}',a'(\tilde{z},a)]+\tilde{d}'\}\big) \\ +\tilde{\mu}_i(\tilde{z},a), \end{aligned} \quad (\mathbf{16})$$

$$\tilde{c}_i(\tilde{z},a) + \tilde{p}(\tilde{z},a)a_i'(\tilde{z},a) = \tilde{y}_i(1-\tilde{d}) + [\tilde{p}(\tilde{z},a)+\tilde{d}]a_i, \quad (\mathbf{17})$$

$$\tilde{c}_1(\tilde{z},a) + \tilde{c}_2(\tilde{z},a) = 1, \quad (\mathbf{18})$$

$$a_1'(\tilde{z},a) + a_2'(\tilde{z},a) = 1, \quad (\mathbf{19})$$

$$\tilde{W}_i(\tilde{z},a) = \frac{\tilde{c}_i(\tilde{z},a)^{1-\gamma}}{1-\gamma} + \beta E \frac{\tilde{W}_i[\tilde{z}',a'(\tilde{z},a)]}{(g')^{\gamma-1}}, \quad (\mathbf{20})$$

$$\tilde{V}_i(\tilde{z}) = \frac{[\tilde{y}_i(1-\tilde{d})]^{1-\gamma}}{1-\gamma} + \beta E \frac{\tilde{V}_i(\tilde{z}')}{(g')^{\gamma-1}}, \quad (\mathbf{21})$$

$$\mu_i(\tilde{z},a)[a_i'(\tilde{z},a) - A_i] = 0, \quad \mu_i(\tilde{z},a) > 0, \quad \text{if } a_i'(\tilde{z},a) = A_i, \quad (\mathbf{22})$$

$$A_i = \max_{\tilde{z}\in\Omega}\{\underline{a}_i : \tilde{W}_i[\tilde{z}, \underline{a}_i(\tilde{z})] = \tilde{V}_i(\tilde{z})\}, \quad (\mathbf{23})$$

where $\tilde{z} = (g, \tilde{d}, \tilde{y}_1, \tilde{y}_2)$, $\tilde{d}$ is the ratio of dividends to total output, $a = (a_1, a_2)$ is the beginning-of-period stockholdings, $\tilde{\mu}_i(\tilde{z},a) = \mu_i(z,a)/Y^{1-\gamma}$ is the normalized Lagrangian multiplier, and variables with primes represent next-period values.

The main idea of the algorithm is as follows. Denote h as the control function that consists of consumption allocation, asset holdings, asset prices, and the Lagrangian multipliers. Denote W as the expected discounted utility. Let $\mathcal{T}$ be the nonlinear operator such that $\mathcal{T}[h \;\; W]'$ satisfies the above equations. Then, an equilibrium control and value function is a fixed point $[h \;\; W]' = \mathcal{T}[h \;\; W]'$ of the nonlinear operator $\mathcal{T}$. We assume that $\mathcal{T}$ exists and is well defined. Then, the nonlinear operator $\mathcal{T}$ can be obtained by the Gauss–Seidel iterations, i.e.,

$$[h_{n+1} \quad W_{n+1}]' = \mathcal{T}[h_n \quad W_n]', \qquad n \geq 0 \tag{24}$$

converges for a given $[h_0 \;\; W_0]'$.

Specifically, the following steps are involved in approximating a fixed point of $\mathcal{T}$. First, use the Hermite–Gauss quadrature rule to discretize the exogenous driving processes to get a finite-state Markov chain along with its transition probability matrix. Tauchen and Hussey (1991) provide details on how to discretize a VAR(p) process into a Markov chain. Second, define a grid

$$\mathcal{D} = (z_j, a_k), \qquad j = 1, 2, \ldots, J; \quad k = 1, 2, \ldots, K.$$

The lower bound on agents' stockholdings is determined when the equilibrium is found. We give a certain initial value to set up the grid. Third, solve a set of linear equations to obtain the values of $V_i(z)$ on the grid $\mathcal{D}$. Given that the exogenous variables take finite states, finding $V_i(z)$ is equivalent to solving the following set of linear equations on V_i:

$$\begin{bmatrix} 1-\beta\pi_{11}g_1 & -\beta\pi_{12}g_2 & \cdots & -\beta\pi_{1J}g_J \\ -\beta\pi_{21}g_1 & 1-\beta\pi_{22}g_2 & \cdots & -\beta\pi_{2J}g_J \\ \vdots & \vdots & \ldots & \vdots \\ -\beta\pi_{J1}g_1 & -\beta\pi_{J2}g_2 & \cdots & 1-\beta\pi_{JJ}g_J \end{bmatrix} \begin{bmatrix} V_i(1) \\ V_i(2) \\ \vdots \\ V_i(J) \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\{\tilde{y}_i(1)[1-\tilde{d}(1)]\}^{1-\gamma}}{1-\gamma} \\ \dfrac{\{\tilde{y}_i(2)[1-\tilde{d}(2)]\}^{1-\gamma}}{1-\gamma} \\ \vdots \\ \dfrac{\{\tilde{y}_i(J)[1-\tilde{d}(J)]\}^{1-\gamma}}{1-\gamma} \end{bmatrix}, \tag{25}$$

where $\{\pi_{ij}\}$, $i, j = 1, 2, \ldots, J$, is the transition probability matrix, and $g_j = [g(j)]^{1-\gamma}$, for $i = 1, 2, \ldots, J$.

Fourth, define the finite-dimensional set $[H_D \;\; W_D]'$ in which a typical element is a function $[h \;\; W]'$ that consists of values on the grid $\mathcal{D}$ along with an interpolation rule[5] to compute the values off the grid. Fifth, given an initial function $[h^0 \;\; W^0]' \in [H_D \;\; W_D]'$, compute $[h^1 \;\; W^1]$ on the grid $\mathcal{D}$ such that

$$\tilde{p}(\tilde{z},a)[\tilde{c}_i(\tilde{z},a)]^{-\gamma} = \beta \sum_{j=1}^{J} \left\{\tilde{c}_i^0[\tilde{z}', a'(\tilde{z},a)]\right\}^{-\gamma} (g')^{-\gamma} \{\tilde{p}^0[\tilde{z}', a'(\tilde{z},a)] + \tilde{d}'\}$$
$$\times \pi(j \mid m) + \tilde{\mu}_i(\tilde{z},a), \tag{26}$$

$$\tilde{W}_i(\tilde{z},a) = \frac{\tilde{c}_i(\tilde{z},a)^{1-\gamma}}{1-\gamma} + \beta \sum_{j=1}^{J} \frac{\tilde{W}_i^0[\tilde{z}', a'(\tilde{z},a)]}{(g')^{\gamma-1}} \pi(j \mid m), \tag{27}$$

and equations (17)–(19) for $m = 1, 2, \ldots, J$. Sixth, use the following formula to find the lower bound on agents' stockholdings:

$$A_i = \max_{j \in \{1,\ldots,J\}} \{\underline{a}_i(j) \colon \tilde{W}_i[\tilde{z}(j), \underline{a}_i(j)] = \tilde{V}_i[\tilde{z}(j)]\}. \tag{28}$$

Use A_i as the lower bound for a_i to update the grid $\mathcal{D}$. Seventh, use $[h^1 \;\; W^1]$ as next $[h^0 \;\; W^0]$ and iterate until $[h \;\; W]$ converges according to some preset criteria.

4.2. Calibration

The calibration involves finding the parameters governing the exogenous driving processes. We use the specification from Heaton and Lucas (1996). Let

$$Z_t = \left[\log\left(\frac{Y_t}{Y_{t-1}}\right), \log\left(\frac{d_t}{Y_t}\right), \log\left(\frac{y_{1t}}{Y_t - d_t}\right)\right],$$

where Y_t and d_t are the total output and dividends at time t, respectively, and y_{1t} is agent 1's endowment. They assume the following VAR specification for the exogenous processes:

$$\begin{bmatrix} Z_{1t} \\ Z_{2t} \\ Z_{3t} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix} \begin{bmatrix} Z_{1t-1} \\ Z_{2t-1} \\ Z_{3t-1} \end{bmatrix} + \begin{bmatrix} E \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} \\ \epsilon_{3t} \end{bmatrix}, \tag{29}$$

where E is a lower triangular matrix.

Heaton and Lucas estimate these parameters using annual aggregate income and dividend data from the National Income and Product Account and annual household income data from the Panel Study of Income Dynamics. The result is reported in Table 1. We then apply the Hermite–Gauss quadrature rule to discretize

TABLE 1. Vector autoregression coefficients for exogenous processes

Variables	Const.	Slopes			Covariance		
Z_1	0.2249	0.0805	0.0626	0.0	0.00076176	0.00015456	0.0
Z_2	−0.1630	−0.5166	0.9480	0.0	0.00015456	0.0018976	0.0
Z_3	−0.33499	0.0	0.0	0.53748	0.0	0.0	0.062901

TABLE 2. Markov chain for exogenous state variables

State No.	States		
	$\log[Y(t)/Y(t-1)]$	$d(t)/Y(t)$	$y_1(t)/[Y(t)-d(t)]$
1	−0.8998%	0.0345	0.3772
2	4.6202%	0.0348	0.3772
3	−0.8998%	0.0376	0.3772
4	4.6202%	0.0380	0.3772
5	−0.8998%	0.0345	0.6228
6	4.6202%	0.0348	0.6228
7	−0.8998%	0.0376	0.6228
8	4.6202%	0.0380	0.6228

TABLE 3. Transition probability matrix

	State No.							
π_{ij}	1	2	3	4	5	6	7	8
1	0.35775	0.24407	0.085432	0.058285	0.12210	0.083305	0.029159	0.019893
2	0.34070	0.33746	0.033850	0.033527	0.11629	0.11518	0.011553	0.011443
3	0.033527	0.033850	0.33746	0.34070	0.011443	0.011553	0.11518	0.11629
4	0.058285	0.085432	0.24407	0.35775	0.019893	0.029159	0.083305	0.12210
5	0.12210	0.083305	0.029159	0.019893	0.35775	0.24407	0.085432	0.058285
6	0.11629	0.11518	0.011553	0.011443	0.34070	0.33746	0.033850	0.033527
7	0.011443	0.011553	0.11518	0.11629	0.033527	0.033850	0.33746	0.34070
8	0.019893	0.029159	0.083305	0.12210	0.058285	0.085432	0.24407	0.35775

TABLE 4. Summary statistics of exogenous variables

	$\log[Y(t)/Y(t-1)]$	$d(t)/Y(t)$	$y_1(t)/Y(t)$
Mean, %	1.9	3.6	50
Std. error, %	2.8	0.16	12

the above VAR. The results are presented in Tables 2 and 3. There are eight states for the discretized exogenous stochastic processes. The high growth rate of output is 4.6% and the low is −0.9% per year. The mean of the output growth rate is 1.9% with an annual standard deviation of 2.8%. Dividends account for 3.6% of total output on average with 0.16% of standard deviation annually. The share of agent 1's production output in total production output has the mean of 50% with a standard deviation of 12% per year (see Table 4).

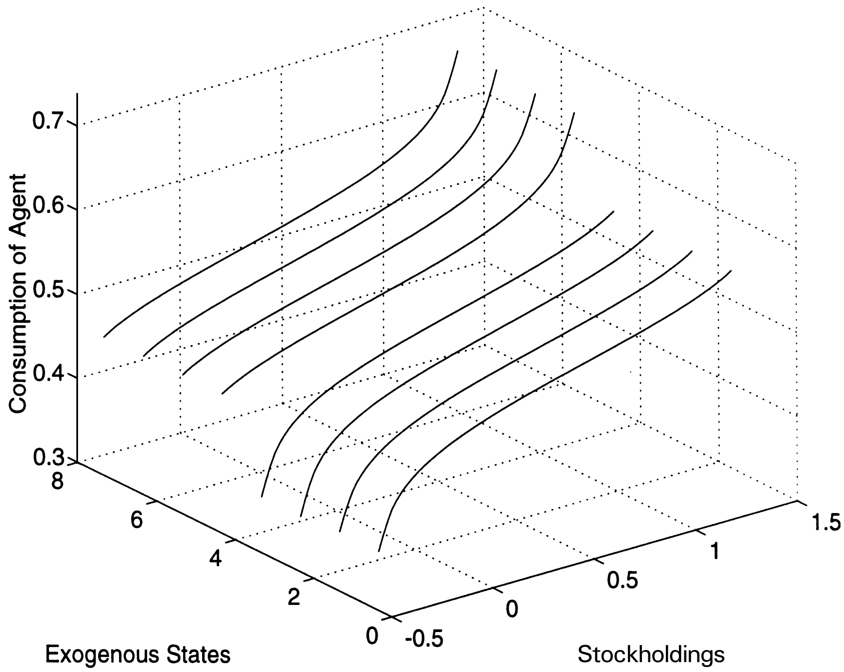

FIGURE 2. Equilibrium consumption function ($\beta = 0.98$, $\gamma = 1.5$, $A = 37\%$).

5. NUMERICAL RESULTS

5.1. Endogenous Short-Sale Limits

Proposition 1 indicates that the endogenous short-sale constraint is a function of agents' risk aversion and time preferences for a given exogenous driving process. To investigate the relationship between the short-sale limits and preference parameters, we solve the equilibrium for the economy with various parameter combinations. For all combinations of risk-aversion coefficients (range from 1.5 to 5.0) and the discount factors (0.97 and 0.98) used in our numerical analysis, the short-sale limits vary from 27 to 45% of total outstanding shares (see Table 5). These results indicate that the commonly used no-short-sale constraint is far more stringent than what is needed to rule out default in equilibrium. Therefore, the difference obtained for an economy with no short sale and a complete-market economy cannot be attributed to ruling out default.

The results show that both risk aversion and time preference affect the limits that agents are allowed to short. For a fixed discount factor, as the risk-aversion coefficient increases, the short-sale limit (in absolute value) first increases and then decreases. This can be attributed to the relative movements of autarky utility (V) and the expected discount utility from trading stocks (W). When risk-aversion coefficients initially increase, both V and W shift upward, but W increases more

TABLE 5. Endogenous short-sale limits

	β	
γ	0.97	0.98
1.5	0.32822	0.37201
2.0	0.37795	0.42553
3.0	0.40631	0.45476
4.0	0.37708	0.40198
5.0	0.27111	0.32398

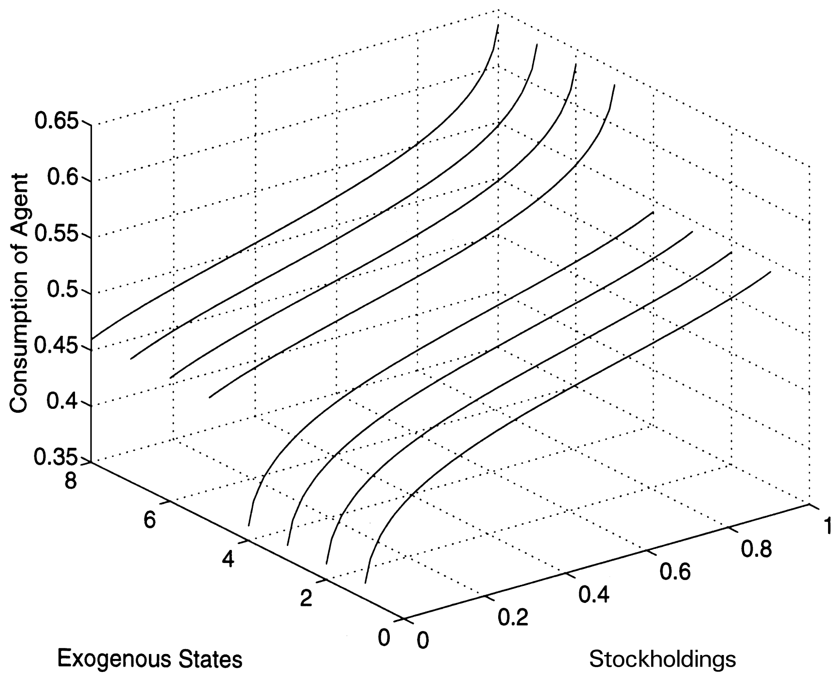

FIGURE 3. Equilibrium consumption function ($\beta = 0.98$, $\gamma = 1.5$, $A = 0$).

than V does. Thus stockholding can take smaller value (larger in absolute value) to equate V and W. This means that agents are allowed to short more. But as agents' risk aversion increases, say from 3.0 to 4.0, both the autarky utility (V) and expected discounted utility (W) start to shift downward. The decrease in W is more than the decrease in V. Hence, stockholdings have to take a larger value (smaller in absolute value) to equate V and W. This implies a tighter short-sale limit.

For a fixed risk-aversion coefficient, as the discount factor increases from 0.97 to 0.98, the short-sale limits also increase (in absolute values). The increase in

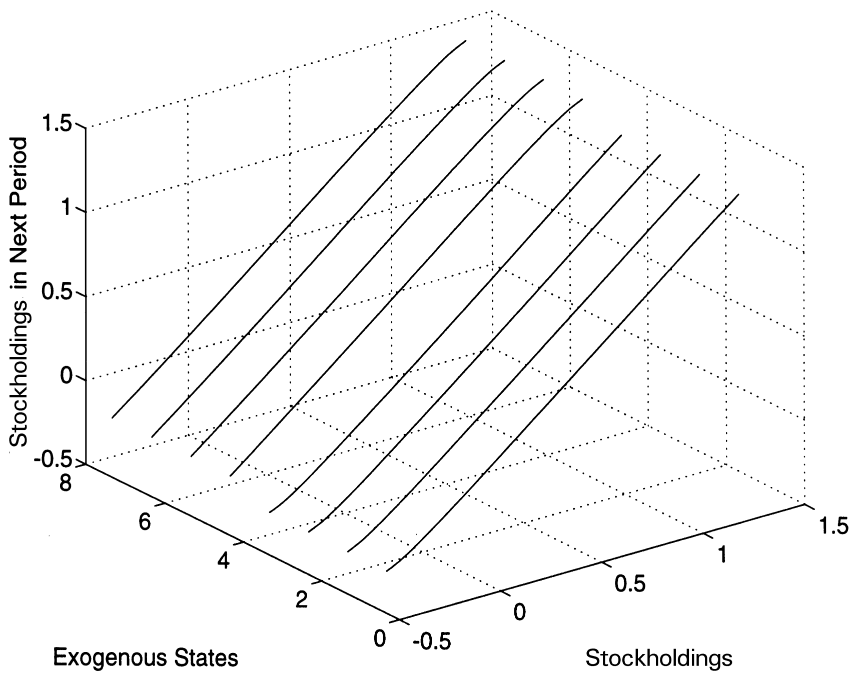

FIGURE 4. Equilibrium stockholding function ($\beta = 0.98$, $\gamma = 1.5$, $A = 37\%$).

the discount factor leads to the downward shift in both V and W. However, the expected discounted utility (W) decreases slower than the autarky utility (V) does. Thus the stockholdings can take a smaller value (larger in absolute value) to equate W and V, which implies a looser short-sale limit.

5.2. Equilibrium Consumption, Stockholdings, Stock Prices, and Stock Trading

In Figures 2–9 we present the equilibrium decision rules for consumption allocation, stockholdings, stock prices, and stock trading as functions of agents' beginning-of-period stockholdings for every exogenous state. The risk-aversion coefficient and utility discount factor are set to 1.5 and 0.98, respectively. For comparison, both decision rules for an economy with no short sale and an economy with the endogenous short-sale constraint (37.2%) are plotted.

For every exogenous state, an agent's consumption for both economies is an increasing function of her beginning-of-period stockholdings. The two functions also exhibit a similar shape. Because the two economies have different short-sale constraints, the domains for the two consumption functions are different. The consumption function for the economy with no short sales is like a squeezed image of that for the economy with short sales. When the beginning-of-period

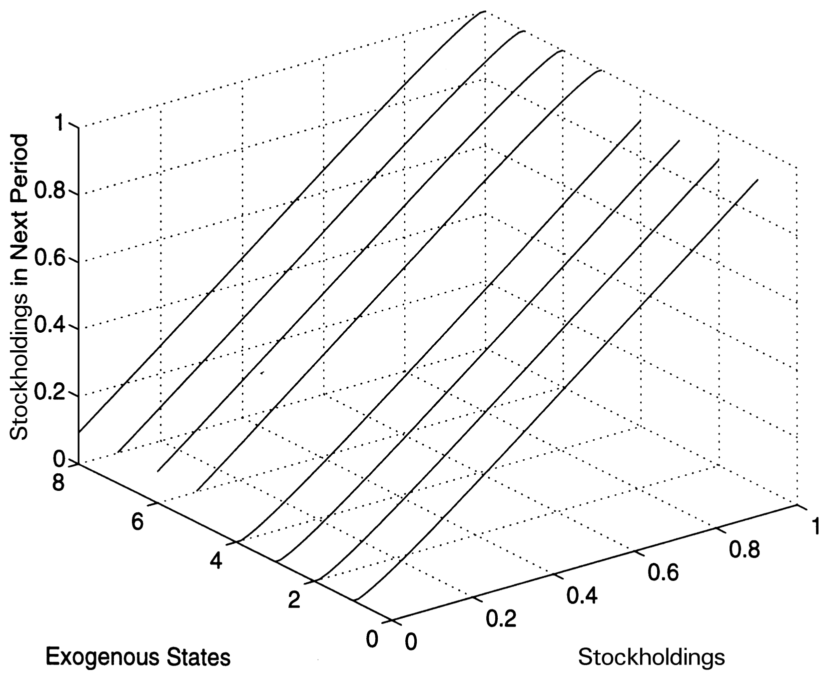

FIGURE 5. Equilibrium stockholding function ($\beta = 0.98$, $\gamma = 1.5$, $A = 0$).

stockholdings take values in the interval [0, 1] the range of consumption function is much wider for the economy with no short sales than for the economy with short sales. This implies that agents in the economy with no short sales will experience more volatile consumption profiles than those in the economy with short sales. The intuition is that allowing agents to sell short increases their ability to smooth their consumption whenever they incur a shock to their output of production. On the other hand, if agents are required to hold nonnegative stocks (i.e., no short sale is allowed), their consumption-smoothing ability will be restricted.

Like consumption, an agent's stockholdings are also increasing functions of her beginning-of-period stockholdings for every exogenous state. They are almost linear for the most part of the domain except for the intervals close to the boundaries where some agents are likely to face a binding short-sale constraint. In the interval [0, 1], the stockholding functions for the two economies are very close to each other except that the range for the one with no short sale is narrower than the one with short sale. This can be explained as follows. Suppose that an agent holds zero stocks at the beginning of the period, in an economy with no short sale, the minimum amount of stocks that she can hold during this period is also zero. If the agent is allowed to sell short, however, her stockholdings this period can be negative if it is optimal for the agent to smooth her consumption. Thus, the range of stockholding function is wider with short sale than without.

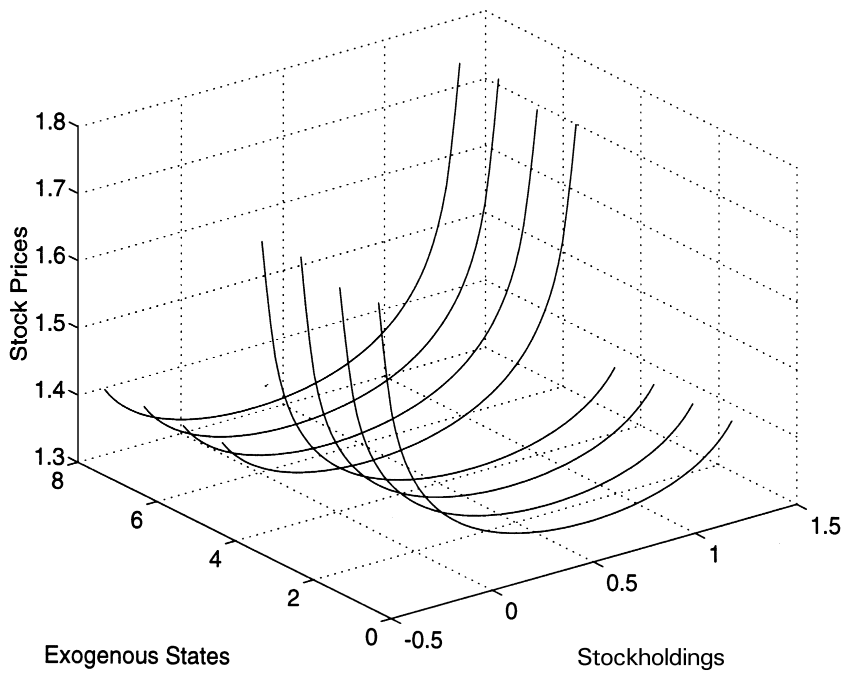

Figure 6. Equilibrium stock-price function ($\beta = 0.98$, $\gamma = 1.5$, $A = 37\%$).

Unlike consumption and stockholdings, stock prices are not monotonic functions of beginning-of-period stockholdings. The shape of stock price functions is more like an asymmetric U. Although stock prices do not change very much when an agent's beginning-of-period stockholdings are away from two end points of their domain, they go up as the stockholdings approach the two end points. This is especially prominent when an agent with a low initial stockholding also has a realization of low output of production. The intuition behind this is that, when agents' stockholdings approach the end points, the wealth distribution among agents can be very uneven. Thus, agents have higher demand to use the stock market to smooth their consumption. Meanwhile, the binding short-sale constraint prevents them from using the market for further consumption smoothing. The agents would value a share of stock more in this situation than in situations in which wealth are evenly distributed among agents. The shape of the price functions has an important economic implication. It indicates that even if asset allocations among agents are, for most of the time, fairly even and their prices do not vary much, rare events such as extremely skewed asset allocations can lead to drastic price changes. We also find that stock-price functions are higher with no short sale than with short sale. This indicates that tighter short-sale constraint has larger impact than looser short-sale constraint.

Stock trading is defined as the change in stockholdings in two adjacent periods, i.e., $|a'(z, a) - a|$. The most important feature of stock trading is that trading

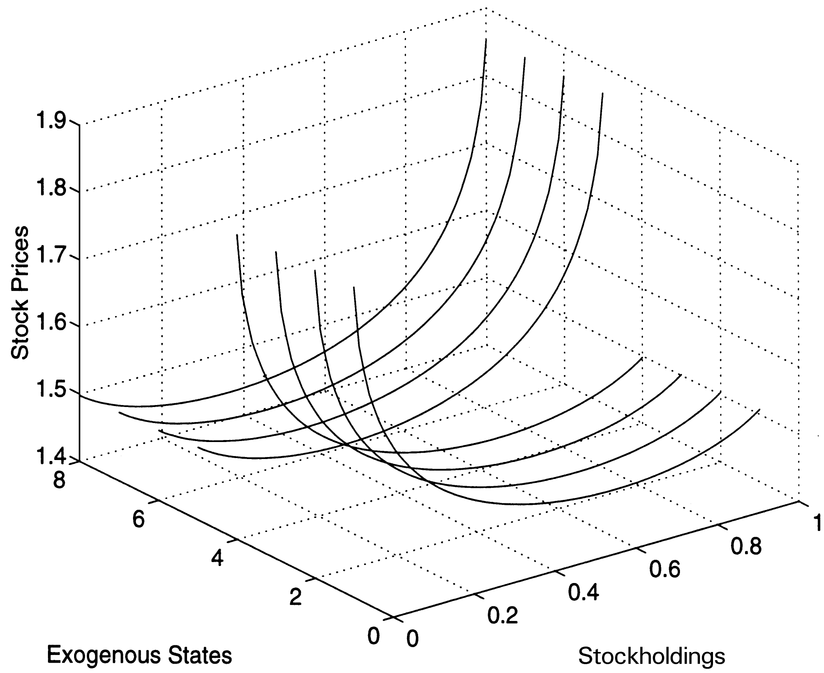

FIGURE 7. Equilibrium stock-price function ($\beta = 0.98$, $\gamma = 1.5$, $A = 0$).

volume is the highest when some agents enter the market with high stock holdings but incur a bad shock to their output of production and some other agents enter the market with low stock holdings but have a good output shock. In this circumstance, the first type of agents sell their stockholdings to the second type in exchange for current consumption good and keep their consumption level from falling too much. Meanwhile, the second type of agents would like to sell their excess consumption good and purchase stocks such that they can do what the first type of agents are doing in case they face the similar situation in the future. Meanwhile, because the consumption good is perishable, they cannot store the excess good as a precaution for the future. The stock-trading functions also show that trading volumes are the lowest when some agents enter the market without much stockholdings and at same time incur a bad shock to their production. Under this circumstance, agents have very high demand for consumption smoothing and value stocks the highest. But their ability to sell short is restricted, they cannot take even shorter positions in their stock holdings. Therefore, stock trading is very inactive.

Given the decision rules, we can calculate the mean and standard deviation of stock returns and trading volume by simulating long series of stock prices and individual stock holdings. Table 6 reports the statistics for three economies: a complete-market economy, an economy with no short sales, and an economy with an endogenous short-sale constraint. The sample stock returns for actual data also are reported as a benchmark.

FIGURE 12. Expected stock-return differences between low- and high-output growth ($\beta = 0.98$, $\gamma = 1.5$): Solid line, short sale; Dashed line, complete markets.

When the beginning-of-period stockholdings are not very skewed between the two types of agents, the impact of the constraint on the countercyclical stock-price movements is very small. However, its impact increases drastically when the beginning-of-period stockholdings approach the two end points. This clearly shows that the countercyclical stock-price movement is related to the presence of the constraint.

The countercyclical stock-price movements also are found to increase as agents become more risk averse. This is uncovered by comparing the differences in stock prices across states for different risk-aversion coefficients. Figure 11 shows the effect of risk-aversion coefficients on differences in stock prices across states. The stock-price differences when the risk-aversion coefficient is 4.0 lie above those when the risk aversion is 1.5. This means that a higher risk-aversion coefficient results in higher countercyclical stock-price movements.

While the stock prices are countercyclical, the expected stock returns implied by the model are procyclical. This is in contrast to the empirical findings of Fama and French (1989). The expected stock return at time t is defined as

$$E_t(r_{t+1}) = E_t[(p_{t+1} + d_{t+1})/p_t]. \quad \mathbf{(30)}$$

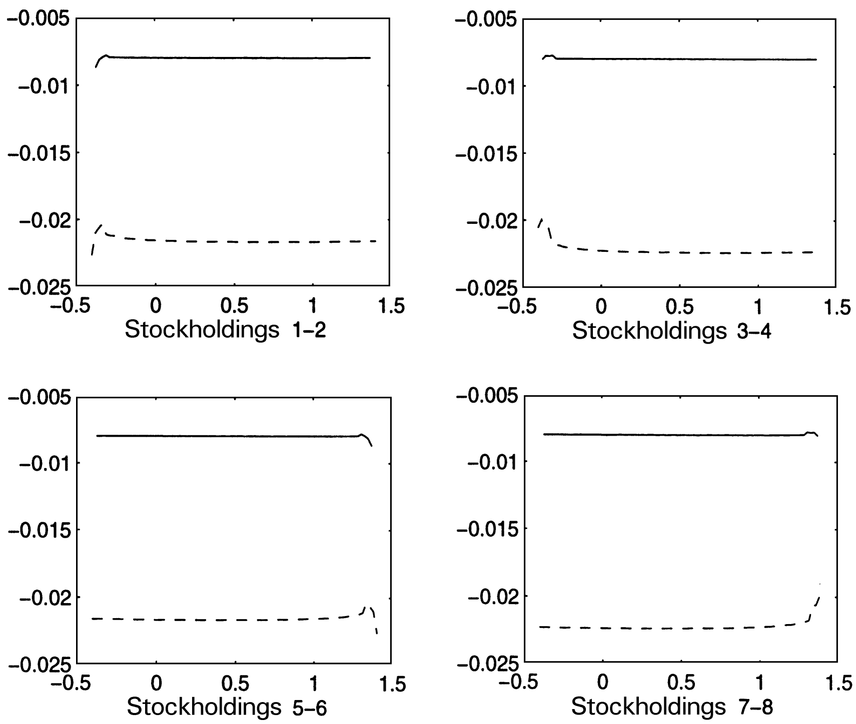

FIGURE 13. Expected stock-return differences between low- and high-output growth ($\beta = 0.98$): Solid line, $\gamma = 1.5$; Dashed line, $\gamma = 4.0$.

The above definition also can be expressed in terms of the normalized stock prices and dividends as follows:

$$E_t(r_{t+1}) = E_t[(\tilde{p}_{t+1} + \tilde{d}_{t+1})g_{t+1}/\tilde{p}_t]. \qquad \textbf{(31)}$$

Denoting the expected stock returns as er_t and rewriting it in state-dependent form yield

$$er(\tilde{z}, a) = E\mathbf{[}(\{\tilde{p}[\tilde{z}', a'(\tilde{z}, a)] + \tilde{d}'\}g'\mathbf{)}/\tilde{p}(\tilde{z}, a)\mathbf{]}, \qquad \textbf{(32)}$$

where the expectation is taken over $\tilde{z}'$. Given the equilibrium decision rules $\tilde{p}(\tilde{z}, a), a'(\tilde{z}, a)$, and the exogenous process for $\tilde{d}$ and g, we can solve for the expected stock-return functions for both the complete- and the incomplete-market economies and examine how the expected returns change for different output growth rates.

Figures 12 and 13 show the relationship between the expected stock returns and the output growth rates for different market structures and structural parameters. In particular, we plot the expected return differentials between low-and high-output growth rates for a complete-market economy and an economy with an endogenous short-sale constraint in Figure 12. It shows that for both markets the expected returns are lower when the output growth rate is lower. In other words, the expected stock returns are procyclical. The expected return differentials between low- and high-output growth rates also increase as individuals' degree of risk aversion increases. This is revealed in Figure 13 in which we plot the expected return differentials for different levels of risk-aversion coefficient.

The above analyses indicate that economies with a time-additive power utility function and exogenous output growth may not be able to explain the relationship between the stock-market movements and business cycles. Time-nonseparable utility functions and physical investment and capital accumulation decisions may be needed to generate the countercyclical expected stock returns.

6. CONCLUSION

In this study we propose an approach to introduce an endogenous constraint on short sale of stocks in an economy in which stock is the only tradable asset. The endogenous short-sale constraint has the interpretation that it is the short-sale limit that an agent won't violate. It is also a function of parameters pertaining to agents' preferences and the exogenous driving forces. Using a numerical algorithm, we find short-sale limits for an array of preference parameters such as risk-aversion coefficient and discount factor in a reasonable range. The short-sale constraints range from 27 to 45% of total outstanding shares.

The mean and standard deviation for the simulated stock returns and trading volume are calculated for the following three economies: a complete-market economy, an economy with no short sales, and an economy with an endogenous short-sale constraint. They then are compared to their observed counterparts from sample data. We find that introducing short-sale constraints improves the performance of economic models in generating the volatility of stock returns. The short-sale constraint alone, however, is not sufficient to explain the puzzles about asset returns. Apparently, more research has to be done to fully understand the roles and extents played by other types of market frictions in addition to short-sale constraints. Further research on the effect of ask-bid spread, transactions costs, and liquidity effects on asset returns is worth pursuing.

The study also reveals that stock prices implied in the economies with short-sale constraints exhibit countercyclical movements. Stock prices are higher when the output growth is low than when it is high. The expected stock returns, on the other hand, are found to be procyclical and are robust across different market structures. This is in contrast to the empirical findings of Fama and French (1989). Possible solutions include adopting time-nonseparable utility functions and introducing individual investment decisions to allow the output growth to be endogenously determined.

Another interesting extension of the study is to include a bond market that allows agents in the economy to borrow and lend. This can approximately confine the motive for short selling to hedging and still allow agents to effectively shift their consumption.[6] The model with both bonds and stocks also allows us to examine the relative price of stocks and bounds, which permits the discussions of risk-free and equity premium puzzles.

NOTES

1. Short interest is the actual number of shares of a stock sold short.
2. In principle, investors can write OTC options on a stock. However, in reality, for many stocks there is no options market because of illiquidity of these options.
3. In practice, a short-sale constraint consists of two aspects: restrictions on the amount a trader can go short and restrictions on the access to the proceeds of her short sale. For simplicity, we assume that agents have full access to the proceeds of their short sales.
4. A more realistic setup is to exclude the person who breaches her contract from intertemporal asset tradings for certain periods and then allow her to come back. This can be incorporated in an overlapping-generations framework.
5. A linear interpolation rule is used in this application.
6. I thank a referee for his suggestion on this issue.

REFERENCES

Aiyagari, S.R. & M. Gertler (1991) Asset returns with transactions costs and uninsured individual risk. *Journal of Monetary Economics* 27, 311–331.

Cambpell, J. & J. Cochrane (1994) By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior. Manuscript, Department of Economics, Harvard University and NBER, and Graduate School of Business, University of Chicago and NBER.

Coleman, J.W. (1990) Solving the stochastic growth model by policy function iteration. *Journal of Business and Economic Statistics* 8, 27–29.

Fama, E. & K. French (1989) Business conditions and expected stock returns. *Journal of Financial Economics* 25, 23–49.

Ferson, W. & J. Merrick (1987) Non-stationarity and stage-of-the-business-cycle effects in consumption-based asset pricing relations. *Journal of Financial Economics* 18, 127–146.

Figlewski, S. (1981) The informational effects of restrictions of short sales: Some empirical evidence. *Journal of Financial and Quantitative Analysis* 16, 463–476.

Figlewski, S. & G. Webb (1993) Options, short sales, and market completeness. *Journal of Finance* 48, 761–777.

Heaton, J. & D. Lucas (1996) Evaluating the effects of incomplete markets on risk sharing and asset pricing. *Journal of Political Economy* 104, 443–487.

Jarrow, R. (1980) Heterogeneous expectations, restrictions on short sales, and equilibrium asset prices. *Journal of Finance* 35, 1105–1113.

Kehoe, T.J. & D.K. Levine (1993) Debt-constrained asset markets. *Review of Economic Studies* 60, 865–888.

Lucas, D.J. (1994) Asset pricing with undiversifiable income risk and short sales constraints: Deepening the equity premium puzzle. *Journal of Monetary Economics* 34, 325–341.

Miller, E. (1977) Risk uncertainty and divergence of opinion. *Journal of Finance* 32, 1151–1168.

Tauchen, G. & R. Hussey (1991) Quadrature-based methods for obtaining approximate solutions to nonlinear asset pricing models. *Econometrica* 59, 371–396.

Telmer, C.I. (1993) Asset pricing and incomplete markets. *Journal of Finance* 48, 1803–1832.

Zhang, H.H (in press) Endogenous borrowing constraints with incomplete markets. *Journal of Finance*.

APPENDIX

Proof of Proposition 1. Agent i's problem at time t is to solve the following Lagrangian function:

$$W_i\left[z_t, a_{i(t-1)}\right] = \max_{c_{is}} E_t\left\{\sum_{s=t}^{\infty}\beta^{s-t}[u(c_{is}) + \mu_{is}(a_{is} - A_i)]\right\},$$

$$c_{is} = y_{is} + (p_s + d_s)a_{i(s-1)} - p_s a_{is}, \qquad \forall\, s \geq t,$$

$$\mu_{is} \geq 0 \quad \text{and} \quad \mu_{is} = 0 \quad \text{iff } a_{is} - A_i \geq 0, \qquad i = 1, 2,$$

where μ_{it} is the time t Lagrangian multiplier of agent i's short-sale constraint. Denote $x_{it} = y_{it} + (p_t + d_t)a_{i(t-1)}$ as agent i's beginning-of-period wealth. Differentiating the indirect utility function with respect to x_{it} and using the envelope theorem, we get:

$$\frac{\partial W_{it}}{\partial x_{it}}\left[z_t, a_{i(t-1)}\right] = u'(c_{it}), \qquad i = 1, 2.$$

Because $u'(c_{it})$ is strictly positive, we have

$$\frac{\partial W_{it}}{\partial x_{it}}\left[z_t, a_{i(t-1)}\right] > 0, \qquad i = 1, 2.$$

By the assumption that stock trading is nontrivial in equilibrium, there exists an $\hat{a}_{i(t-1)}$ such that $W_{it}[z_t, \hat{a}_{i(t-1)}] \geq V_i(z_t)$. Because

$$\frac{\partial W_{it}}{\partial x_{it}}\left[z_t, a_{i(t-1)}\right] > 0 \quad \text{and} \quad \lim_{a_{i(t-1)}\to-\infty} W_{it}[z_t, a_{i(t-1)}] = -\infty,$$

given that $V_i(z_t)$ is finite, there must exist an $\tilde{a}_{i(t-1)}$ such that $W_{it}[z_t, \tilde{a}_{i(t-1)}] \leq V_i(z_t)$. According to the intermediate-value theorem, there exists an $\underset{\sim}{a}_{i(t-1)}$, $\tilde{a}_{i(t-1)} \leq \underset{\sim}{a}_{i(t-1)} \leq \hat{a}_{i(t-1)}$ such that $W_{it}[z_t, \underset{\sim}{a}_{i(t-1)}] = V_i(z_t)$ and $W_{it}[z_t, a_{i(t-1)}] \geq V_i(z_t)$ if and only if $a_{i(t-1)} \geq \underset{\sim}{a}_{i(t-1)}$.

Given that z_t takes a finite number of outcomes, denote $A_i = \max_{z_t\in\Omega_Z}\{\underset{\sim}{a}_i(z_t)\}$, then for any a_i such that $W_{it}(z_t, a_i) \geq V_i(z_t)$ for all z_t, we have

$$a_i \geq A_i.$$

The above condition implies that A_i is the lower bound on a_i.

To prove that A_i is a short-sale limit, we only need to show that $A_i \leq 0$. At any time t, if $a_{i(t-1)} = 0$, then $c_{i\tau} = y_{i\tau}$ and $a_{i\tau} = 0, \forall\, \tau \geq t$ satisfy both the budget constraint and $W_{it}(z_t, 0) \geq V_{it}(z_t)$. Hence, $W_{it}(z_t, 0) \geq V_{it}(z_t)$. Because $W_{it}(z_t, A_i) = V_{it}(z_t)$ and $W_{it}(z_t, .)$ is an increasing function, it must be true that $A_i \leq 0$. Thus, A_i is a short-sale limit. ■