The Continuous Spectrum of Periodically Stationary Pulses in a Stretched Pulse Laser

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A spectral method for determining the stability of periodically stationary pulses in fiber lasers is introduced. Pulse stability is characterized in terms of the spectrum (eigenvalues) of the monodromy operator, which is the linearization of the round trip operator about a periodically stationary pulse. A formula for the continuous (essential) spectrum of the monodromy operator is presented, which quantifies the growth and decay of continuous waves far from the pulse. The formula is verified by comparison with a fully numeric method for an experimental fiber laser. Finally, the effect of the saturable absorber on pulse stability is demonstrated. © 2022 Optica Publishing Group

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1. INTRODUCTION

Since the advent of the soliton laser, researchers have invented several generations of short pulse fiber lasers for a variety of applications, including stretched-pulse (dispersion-managed) lasers [1, 2], similariton lasers [3, 4], and the Mamyshev oscillator [5–7]. A significant modeling challenge is that from one generation to the next there has been a dramatic increase in the amount by which the pulse varies over each round trip. As is highlighted in the survey paper of Turitsyn et al. [8], the computational modeling of modern short pulse lasers should therefore be based on lumped models in which the pulse changes shape as it propagates through the various components of the laser system, returning to the same shape once per round trip. We call such pulses periodically stationary pulses to distinguish them from the stationary pulses in a soliton laser.

A key design issue is to determine those regions in parameter space in which the laser operates stably, and within that space to optimize the pulse parameters. In [8, 9], the authors develop a lumped laser model based on the Haus master equation with dispersion management and discrete dissipative elements. Using a chirped Gaussian pulse ansatz, they derive a reduced system of ordinary differential equations for the peak power, pulse width, and chirp, which they solve numerically to obtain periodically stationary pulses. This reduced method enables rapid exploration of the parameter space of the laser system.

Menyuk and Wang [10] describe the evolutionary and dynamical approaches for determining pulse stability. In the evolutionary approach, an initial pulse is propagated many round trips of the laser. If the pulse converges to a stationary or periodically stationary pulse it is deemed to be stable. For the application of the dynamical approach to stationary solutions of averaged models [10–12], a numerical root finding method is first used to discover a stationary pulse. The stability of this pulse is then determined by computing the spectrum (set of eigenvalues) of the linearization of the differential equation about the pulse.

Cuevas-Maraver et al. [13] used the dynamical approach to determine the stability of the Kuznetsov-Ma breather, which is a periodically stationary analytical solution of the nonlinear Schrödinger equation. By approximating the nonlinear Schrödinger equation as a system of ordinary differential equations, Floquet theory [14] can be used to assess the stability of a periodic solution, ψ, in terms of the spectrum of the monodromy matrix. The monodromy matrix is the matrix, $\mathbf{M}$, for which the solution after one period of the linearization of the differential equation is obtained by multiplying an initial vector, $\mathbf{u}_0$, by $\mathbf{M}$. Eigenvectors of $\mathbf{M}$ whose eigenvalues lie outside the unit circle in the complex plane correspond to perturbations of ψ that grow from one period to the next. Tsoy et al. [15] found periodically stationary solutions of the cubic-quintic complex Ginzburg-Landau equation, which is an averaged model for pulse propagation in a fiber laser. Recently, Zweck et al. [16] derived a formula for the continuous (or essential) spectrum of the monodromy operator for these pulses. The continuous spectrum, which typically consists of a pair of continuous curves in the complex plane, quantifies the growth or decay of perturbations seeded by continuous waves far from the pulse, in contrast to the discrete spectrum that quantifies the growth and decay of perturbations that distort the modelocked pulse.

As the number of points used to discretize the pulse and the width of the time window both increase, the set of eigenvalues of the monodromy matrix converges to the spectrum of the monodromy operator. However, the rate of convergence is often very poor [11], particularly for the continuous spectrum since very large time windows are needed due to the slow decay of the corresponding eigenfunctions. However, since the pulse only affects the continuous spectrum through its contribution to the gain and loss via slow saturation, the continuous spectrum is relatively easy to compute, particularly for stationary pulses [11].

Although the dynamical approach has been extensively applied to stationary pulse solutions of averaged equations [10–

12, 17–19], the results in this paper are the first application of the dynamical approach to periodically stationary pulse solutions of a lumped laser model. In particular, we build upon the results of Zweck et al. [16] by deriving a formula for the continuous spectrum of the monodromy operator of a periodically stationary pulse in a lumped model of a fiber laser. Because the full spectrum consists of the continuous spectrum together with only a few discrete eigenvalues, this result is a significant step towards determining the stability of periodically stationary pulses. The main advantage of our approach compared to that of Turitsyn et al [8, 9] is that changes in the spectrum of the monodromy operator provide insight into the physical mechanisms by which the pulse becomes unstable. Furthermore, as in soliton perturbation theory [20] and in the analysis of wake modes in SESAM fiber lasers [18], we anticipate that the eigenfunctions of the monodromy operator could be used to quantify the effects of noise without the need for Monte Carlo simulations.

2. FIBER LASER MODEL

We consider a lumped model of a fiber laser that is based on a stretched pulse laser of Kim et al. [2]. Although our results are formulated for this specific laser, the formula we present below for the continuous spectrum can be readily adapted to a wide range of short pulse fiber lasers. As we see in the system diagram in Fig. 1, the components in the lumped model are a saturable absorber (SA), two segments of single mode fiber (SMF1 and SMF2), a fiber amplifier (FA), a dispersion compensation element (DCF), and an output coupler (OC), arranged in a loop.

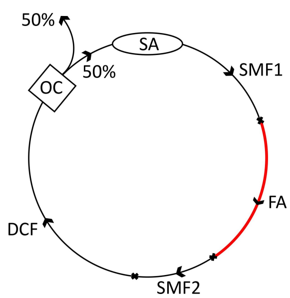

Fig. 1. System diagram of the stretched pulse laser described in [2] that we use for the results in sections 3 and 4.

We let $\psi = \psi(z,t)$ denote the complex electric field envelope of the pulse at position, z, in the loop and time, t, across the pulse. We let $\mathcal{P}$ denote the transfer function of a component, so that $\psi_{\text{out}} = \mathcal{P}\psi_{\text{in}}$, where ψ_{in} and ψ_{out} are the pulses entering and exiting the component. The round trip operator,

$$\mathcal{R} = \mathcal{P}^{\text{OC}} \circ \mathcal{P}^{\text{DCF}} \circ \mathcal{P}^{\text{SMF2}} \circ \mathcal{P}^{\text{FA}} \circ \mathcal{P}^{\text{SMF1}} \circ \mathcal{P}^{\text{SA}}, \tag{1}$$

is the composition of the transfer functions of the components.

We model the transfer function, $\mathcal{P}^{\text{FA}}$, of a fiber amplifier of length, L_{FA}, as $\psi_{\text{out}} = \mathcal{P}^{\text{FA}}\psi_{\text{in}}$, where $\psi_{\text{out}} = \psi(\text{L}_{\text{FA}}, t)$ is obtained by solving the initial value problem

$$\begin{gathered} \partial_z \psi = \left[\frac{g(z)}{2}\left(1 + \frac{1}{\Omega_g^2}\partial_t^2\right) - \tfrac{i}{2}\beta\partial_t^2 + i\gamma|\psi|^2 \right]\psi, \\ \psi(0,t) = \psi_{\text{in}}(t). \end{gathered} \tag{2}$$

Here, $g(z)$ is the saturable gain given by

$$g(\psi) = g_0/(1 + E(z)/E_{\text{sat}}) \tag{3}$$

where, g_0 is the unsaturated gain, $E(z)$ is the pulse energy at position, z, and E_{sat} is the saturation energy. The finite bandwidth of the amplifier is modeled using a Gaussian filter with bandwidth, Ω_g. We model the transfer function, $\mathcal{P}^{\text{SMF}}$, of a segment of single mode fiber of length L_{SMF} by substituting $g_0 = 0$ in Eq. (3). Similarly, $\mathcal{P}^{\text{DCF}}$, is obtained by settting g_0 and $\gamma = 0$.

We model the saturable absorber using the fast saturable loss transfer function [17], $\mathcal{P}^{\text{SA}}$, given by

$$\psi_{\text{out}} = \mathcal{P}^{\text{SA}}(\psi_{\text{in}}) = \left(1 - \frac{\ell_0}{1 + |\psi_{\text{in}}|^2/P_{\text{sat}}}\right)\psi_{\text{in}}, \tag{4}$$

where ℓ_0 is the unsaturated loss and P_{sat} is the saturation power. With this model, ψ_{out} at time t only depends on ψ_{in} at the same time, t. Finally, we model the transfer function, $\mathcal{P}^{\text{OC}}$, of the output coupler as

$$\psi_{\text{out}} = \mathcal{P}^{\text{OC}}\psi_{\text{in}} = \sqrt{L_{\text{OC}}}\psi_{\text{in}}, \tag{5}$$

where L_{OC} is the power loss at the output coupler.

The parameters in the model are similar to those in the experimental stretched pulse laser of Kim [2]. The parameters for the saturable absorber are given in section 4. The saturable aborber is followed by a segment of single mode fiber with $\gamma = 2 \times 10^{-3}$ (Wm)$^{-1}$, $\beta = 10$ kfs^2/m, and $L = 0.32$ m, a fiber amplifier with $g_0 = 6\text{m}^{-1}$, $E_{\text{sat}} = 200$ pJ, $\Omega_g = 50$ THz, $\gamma = 4.4 \times 10^{-3}$ (Wm)$^{-1}$, $\beta = 25$ kfs^2/m, and $L = 0.22$ m, a second segment of SMF with $L = 0.11$ m, a dispersion compensation element with $\beta = -1$ kfs^2, and a 50% output coupler.

A periodically stationary pulse is a pulse, $\psi^{(0)}$, for which

$$\mathcal{R}\psi^{(0)} = e^{i\theta}\psi^{(0)}, \tag{6}$$

for some constant phase, θ. We formulate the problem of discovering a periodically stationary pulse as that of finding a zero of the residual,

$$\mathcal{E}(\psi^{(0)}, \theta) = \frac{1}{2}\int_{-\infty}^{\infty} |\mathcal{R}\psi^{(0)}(t) - \text{e}^{i\theta}\psi^{(0)}(t)|^2 dt. \tag{7}$$

Since $\mathcal{E} \geq 0$, in practice we minimize $\mathcal{E}$ with respect to $\psi^{(0)}$ and θ using a gradient-based iterative optimization method.

To assess the stability of a periodically stationary pulse, ψ, in the presence of perturbations, we linearize the round trip operator, $\mathcal{R}$, about ψ. Because the linearization of Eq. (2) involves the complex conjugate of ψ, we reformulate the equations in the model as a system of equations for a real column vector $\boldsymbol{\psi} = [\Re(\psi), \Im(\psi)]^T = [\psi_1, \psi_2]^T$. We linearize the transfer function of each component using a perturbation $\boldsymbol{\psi}_\epsilon = \boldsymbol{\psi} + \epsilon\mathbf{u}$ about $\boldsymbol{\psi}$. For example, for a single mode fiber segment, the linearized transfer function is given by $\mathbf{u}_{\text{out}} = \mathcal{U}^{\text{SMF}}\mathbf{u}_{\text{in}}$, where $\mathbf{u}_{\text{out}} = \mathbf{u}(\text{L}_{\text{SMF}}, t)$ is obtained by solving the linearized initial value problem[1]

$$\begin{gathered} \partial_z \mathbf{u} = \left(\mathbf{L} + \mathbf{M}_1(\boldsymbol{\psi}) + \mathbf{M}_2(\boldsymbol{\psi})\right)\mathbf{u}, \\ \mathbf{u}(0,t) = \mathbf{u}_{\text{in}}, \end{gathered} \tag{8}$$

where

$$\mathbf{L} = -\frac{\beta}{2}\mathbf{J}\partial_t^2, \quad \mathbf{M}_1(\boldsymbol{\psi}) = \gamma(\psi_1^2 + \psi_2^2)\mathbf{J}, \quad \mathbf{M}_2(\boldsymbol{\psi}) = 2\gamma\mathbf{J}\boldsymbol{\psi}\boldsymbol{\psi}^T,$$

with $\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The linearization of the round trip operator, $\mathcal{R}$, about $\boldsymbol{\psi}$ determines whether or not a perturbation, $\mathbf{u}$, grows

[1] See Supplement 1 for the derivation of this equation, and for formulae for the linearizations of the other transfer functions in the model.

from round trip to round trip. This linear operator, which is called the monodromy operator for $\boldsymbol{\psi}$, is given by

$$\mathcal{M} = \mathcal{U}^{\text{OC}} \circ \mathcal{U}^{\text{DCF}} \circ \mathcal{U}^{\text{SMF2}} \circ \mathcal{U}^{\text{FA}} \circ \mathcal{U}^{\text{SMF1}} \circ \mathcal{U}^{\text{SA}}, \tag{9}$$

where each operator $\mathcal{U}$ is the linearization of the corresponding transfer function $\mathcal{P}$ about $\boldsymbol{\psi}$. A fundamental result of the Floquet theory of periodic differential equations asserts that a periodic pulse is stable if the spectrum of $\mathcal{M}$ lies on or inside the unit disc in the complex plane, and that $\boldsymbol{\psi}$ is unstable if at least one eigenvalue of $\mathcal{M}$ lies outside the unit disc [14].

3. FORMULA FOR THE CONTINUOUS SPECTRUM

The spectrum of a linear operator, $\mathcal{M}$, on a function space consists of the set of eigenvalues together with the continuous (essential) spectrum, $\sigma_{\text{cont}}(\mathcal{M})$. Since the continuous spectrum quantifies the growth rate of continuous waves far from the pulse, it is given in terms of the asymptotic monodromy operator,

$$\mathcal{M}_\infty = \mathcal{U}^{\text{OC}}_\infty \circ \mathcal{U}^{\text{DCF}}_\infty \circ \mathcal{U}^{\text{SMF2}}_\infty \circ \mathcal{U}^{\text{FA}}_\infty \circ \mathcal{U}^{\text{SMF1}}_\infty \circ \mathcal{U}^{\text{SA}}_\infty, \tag{10}$$

where, for each component, $\mathcal{U}_\infty$ is obtained by setting $\boldsymbol{\psi} = 0$ in the corresponding operator $\mathcal{U}$.

The continuous spectrum of $\mathcal{M}_\infty$ consists of all complex λ for which there is a real μ and a nonzero $\mathbf{w} \in \mathbb{C}^2$ so that $\mathcal{M}_\infty e^{i\mu t}\mathbf{w} = \lambda e^{i\mu t}\mathbf{w}$. For the lumped model of the laser shown in Fig. 1, the set of all such $\lambda = \lambda(\mu)$ consists of the complex conjugate pair of curves parameterized by

$$\lambda_\pm(\mu) = \frac{1}{\sqrt{2}}(1-\ell_0) \times \exp\left\{\frac{1}{2}\left(1-\frac{\mu^2}{\Omega_g^2}\right)\int_0^{L_{\text{FA}}} g(z)dz\right\} \exp\left\{\pm i\frac{\mu^2}{2}\beta_{\text{RT}}\right\}, \tag{11}$$

for all $\mu \in \mathbb{R}$. Here β_{RT} is the round trip dispersion. In this case, the continuous spectrum consists of a pair of counter-rotating spirals whose amplitudes decay to zero as fast as a Gaussian decays to zero as $\mu \to \pm\infty$.

To ensure that a continuous wave with frequency μ experiences more loss than gain far from the pulse, we require that $|\lambda_\pm(\mu)| \leq 1$, which holds for all μ provided that

$$\frac{1}{2}(1-\ell_0)^2 G^{\text{FA}}_{\text{Tot}} \leq 1, \tag{12}$$

where

$$G^{\text{FA}}_{\text{Tot}} = \exp\left\{\int_0^{L_{\text{FA}}} g(z)\, dz\right\}. \tag{13}$$

If the root mean square width of the power spectral density of the pulse is sufficiently smaller than the gain bandwidth, then $G^{\text{FA}}_{\text{Tot}} \approx E^{\text{FA}}_{\text{out}}/E^{\text{FA}}_{\text{in}}$ is approximately the energy gain in the fiber amplifier. Eq. (12) is a necessary condition for stability since it ensures that continuous waves do not grow. Slow saturable gain plays a critical role in determining pulse stability since the gain experienced by the continuous waves decreases when the pulse energy increases. We note that because the loss saturates at high power, the gain and loss can still balance for the pulse. Consequently, the saturable absorber also plays a critical role in stabilizing the system [17]. While Eq. (11) is specific to the model considered here, the basic approach of taking into account the impact of the pulse on the gain and loss experienced by continuous waves can be readily adapted to any ultrafast laser.

For an idealized soliton laser, continuous waves do not grow since the continuous spectrum lies entirely on the imaginary axis, rather than extending into the right half of the complex plane [20]. For periodically stationary solutions of the constant coefficient Ginzburg-Landau equation [16], the continuous spectrum lies in the open unit disc provided that the loss parameter of the equation is positive (i.e., linear loss exceeds linear gain). In both cases, the stability of continuous waves is independent of all other system parameters. However, for periodically stationary solutions in a fiber laser that includes a fiber amplifier with bandlimited gain saturation, the continuous spectrum, can depend in a complex way on the interplay between all the system parameters, since they all influence the shape of the pulse and hence the total gain, $G^{\text{FA}}_{\text{Tot}}$, in the fiber amplifier.

4. NUMERICAL RESULTS

In this section, we use Eq. (11) to calculate the continuous spectrum of the monodromy operator for periodically stationary pulses in a lumped model of an experimental femotsecond fiber laser. In particular, we demonstrate the effect that the parameters in the fast saturable absorber have on the continuous spectrum.

To compute periodically stationary pulses, we use an N-point discretization of the time window, which yields a discretized pulse $\boldsymbol{\psi} = (\Re(\psi), \Im(\psi))$ in a $2N$-dimensional Euclidean space, $\mathbb{R}^{2N}$. We then optimize the residual $\mathcal{E}$, which is a real-valued function on $\mathbb{R}^{2N+1}$, using the Broyden–Fletcher–Goldfarb–Shanno (BFGS) optimization algorithm [21]. To calculate the gradient of $\mathcal{E}$ we adapted a method of Ambrose and Wilkening [22]. With this method, the cost of computing a directional derivative of $\mathcal{E}$ is comparable to that of propagating a pulse and its linearization for one round trip through the laser. To obtain an initial guess for the optimization method we propagate a Gaussian pulse for a few round trips. Then we use optimization to drive the residual, $\mathcal{E}$, very close to zero, thereby obtaining a periodically stationary pulse. Once we have obtained a periodically stationary pulse, we verify Eq. (11) by computing the eigenvalues of the $2N \times 2N$ matrix discretization, $\mathbf{M}$, of the monodromy operator, $\mathcal{M}$. The computation of each column of $\mathbf{M}$ involves solving an initial value problem for the linearized system for one round trip through the laser [13].

We consider two sets of parameters for the saturable absorber in the lumped model described in section 2. For parameter set 1, we choose $\ell_0 = 0.1$ and $P_{\text{sat}} = 2000$ W (weak saturation) and for parameter set 2, $\ell_0 = 0.2$ and $P_{\text{sat}} = 50$ W (strong saturation). In the far left panel of Fig. 2 we show the amplitude of the periodically stationary pulse at the output coupler as computed using the evolutionary method (solid blue) and the optimization method (dashed red). We observe that the wings on the sides of the solid blue pulse grow as the number of round trips increases in the evolutionary method, which suggests that the pulse may be unstable. In the center left panel, we show the set of all eigenvalues of the discretized monodromy matrix, $\mathbf{M}$, (blue circles) and the continuous spectrum, $\sigma_{\text{cont}}(\mathcal{M})$, obtained using Eq. (11) (solid red), both computed for the optimized pulse in the far left panel. The excellent agreement between the red and blue counter-rotating spirals provides a strong verification of Eq. (11). For this laser, the continuous spectrum of the monodromy operator is well approximated by a set of eigenvalues of $\mathbf{M}$, which coalesce to the continuous spectrum of $\mathcal{M}$ as N increases. The spectral method shows that the periodically stationary pulse is unstable, since $\sigma_{\text{cont}}(\mathcal{M})$ extends outside the unit circle near $\lambda = 1$. The largest eigenvalue on the real axis is at $\lambda = 1.0151$ using Eq. (11) and at $\lambda = 1.015$ using the numerical computation of the eigenvalues of $\mathbf{M}$. Indeed, if we evolve the optimized pulse for many round trips, low-frequency perturbations grow,

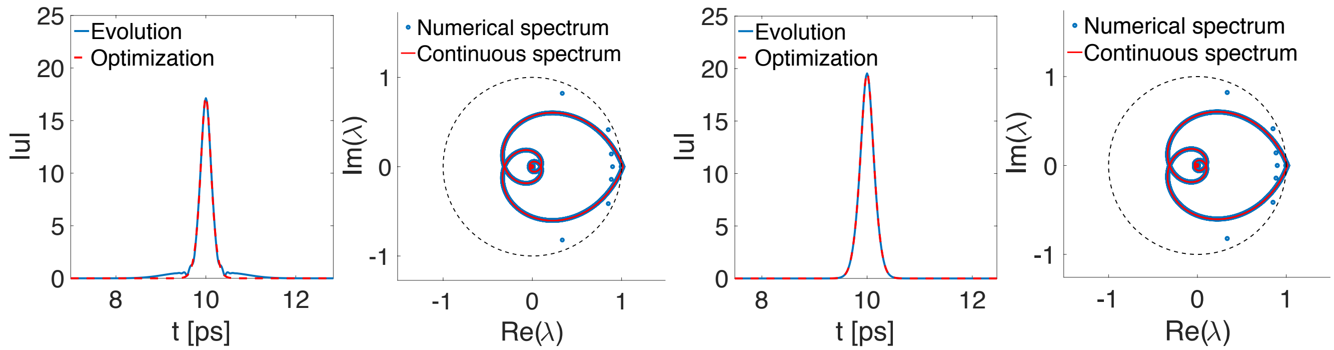

Fig. 2. Far left: Periodically stationary pulse obtained using the evolutionary approach (solid blue) and optimization (dashed red) for parameter set 1. Center left: Eigenvalues of the discretized monodromy matrix, **M**, (blue circles) and the continuous spectrum, $\sigma_{\text{cont}}(\mathcal{M})$, computed using Eq. (11) (solid red). Center and far right: Corresponding results for parameter set 2.

generating a rising floor. These perturbations correspond to the small portion of the continuous spectrum that lies outside the unit circle. Physically, this result is to be expected since P_{sat} is much larger than the peak power of the pulse.

In the center and far right panels we show the corresponding results for parameter set 2. In this case, the pulse obtained using the evolutionary approach (solid blue), agrees very well with the optimized pulse (dashed red). More significantly, the spectrum of the optimized pulse lies inside the unit circle, which shows that this pulse is indeed stable. In this case, the largest eigenvalue on the real axis is at $\lambda = 0.8554$ using Eq. (11) and at $\lambda = 0.85538$ using the numerical method. We confirmed this conclusion by propagating the optimized pulse for many round trips. By increasing the unsaturated loss, ℓ_0, in the saturable absorber so that Eq. (12) holds for parameter set 2, we have ensured that the growth of continuous waves is suppressed. Furthermore, by decreasing the saturation power, P_{sat}, to a value that is comparable to the pulse power, the pulse experiences less loss than does the continuum. As a consequence, the saturable gain effect in the fiber amplifier enables the system to balance the gain and loss of the pulse, which results in a pulse that is periodically stationary.[2]

Already in the 1975, Haus [23] identified the need for a saturable absorber to suppress the growth of continuous waves, while balancing gain and loss for the pulse. The importance of our work is to introduce a method that enables modelers to accurately quantify these effects in lasers for which there is significant pulse breathing. This full model approach complements previous reduced model approaches [8, 9] for determining regions in parameter space that support periodically stationary pulses. By computing the spectrum of the monodromy operator, we can rigorously assess the linear stability of periodically stationary pulses, identify the physical mechanisms that lead to instability, and ultimately quantify the effects of noise [10, 19].

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Data availability. Data generated for the results presented in this paper may be obtained from the authors upon request.

Supplemental document. See Supplement 1 for supporting content.

[2]See Supplement 1 for additional simulation results.

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