Math 2415

Problem Section #4

Make sure you do some problems from each section.

12.6: Cylinders and Quadric Surfaces

We recommend that each of you attempts to sketch the curves and surfaces below for yourself on paper. Our goal is not to become great artists (though that would be nice). Rather the sketches are supposed to help you construct an accurate visual representation of the curve or surface in your mind. You can share photos/scans of your sketches with each other. As you are sketching you can explain to each other what you are doing, how you are doing it, and why it is a good thing to do!

1. In this problem we will make a picture of the surface $y = x^2$ in $\mathbf{R}^3$ as follows. The approach below can be used to plot any surface in space whose equation is independent of one of the variables, (x, y, z).
   (a) In $\mathbb{R}^2$ the equation $y = x^2$ represents a curve in the plane. Sketch it.
   (b) However, the set of points (x, y, z) in $\mathbf{R}^3$ for which $y = x^2$ represents a surface in space. Why do we expect an equation of the form $F(x, y, z) = 0$ to be a surface (i.e., a curved sheet) in space? [**Hint:** Imagine you could manipulate the equation $F(x, y, z) = 0$ to express one of the variables as a function of the other two.] For $y = x^2$ what is the function F?
   (c) To visualize this surface, translate the curve your sketched in (a) up and down along the z-axis to form a surface.
   (d) What would you need to do to make a picture of the surface $x^2 + 4z^2 = 1$?

2. In this problem we will visualize the surface $z^2 - x^2 - y^2 = 4$ in space. The approach below can be used to plot a surface in space whenever x and y always appear in the combination $x^2 + y^2$ in the equation.
   (a) First check that when you convert the equation $z^2 - x^2 - y^2 = 4$ to cylindrical coordinates you get $z^2 - r^2 = 4$.
   (b) Plot the curve $z^2 - r^2 = 4$ in the (r, z)-plane.
   (c) Rotate this curve about the z-axis to obtain a surface in space.
   (d) Explain why the resulting surface is the set of points (x, y, z) in space so that $z^2 - x^2 - y^2 = 4$.
   (e) What would you need to do to make a picture of the surface $x^2 + z^2 = y^2$?

3. Make a labelled sketch of the traces (slices) of the surface

$$x^2 - y^2 + 4z^2 = 0$$

   in the planes $x = 0$, $z = 0$, and $y = k$ for $k = 0, \pm 1, \pm 2$. Then make a labelled sketch of the surface. *What do we mean by labelled?* Label the slice you are sketching (e.g. $x = 0$ or

$y = k$), label all axes (e.g. x or z), indicate values of intercepts and other significant points such as vertices, add asymptotes as appropriate, and when graphing multiple curves on the same plot label them with their k-value. Transfer as much of this information as you can to the sketch of the surface in 3-dimensional space. See the examples in Dr. Zweck's lecture notes: Lecture 6A (How to Sketch Quadric Surfaces) and Lecture 6B (Quadric Surfaces)

4. Use the approach in the previous problem to sketch $y = 4x^2 + z^2$

5. Describe and sketch the surface $z = xy$. How is it related to the surface $z = x^2 - y^2$?

13.1: Parametrized Curves

1. Let C be the curve in the xy-plane given by $x = 2\sin t$, $y = -\cos t$. Find an equation of the form $f(x, y) = 0$ for this curve. Sketch the curve. What is the starting point (where $t = 0$)? What direction does the parametrization go along the curve?

2. Describe the motion given by the parametrized curve $x = \cos t$, $y = -\cos t$ for $0 \leq t \leq 3\pi$. You could draw on a white board (make a video if you like!) and provide an accompanying verbal/written explanation.

3. Consider the parametrized curve $\mathbf{r}(t) = \sin t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos t\,\mathbf{k}$ for $0 \leq t \leq \pi/2$.

   (a) Use algebra to find a cylinder on which this curves lies (there are actually two of them).

   (b) Use algebra to find a plane on which this curve lies.

   (c) Use the fact the curve lies on both the plane and cylinder to sketch it. (Note the limited range of t values!)

4. Sketch the curve $x = t\cos t$, $y = t$, $z = 2t\sin t$, for $0 \leq t \leq 2\pi$. To do so, first show that this curve lies on an elliptical cone aligned with the y-axis. Next explain why shadow of the curve on the xz-plane is a counter-clockwise spiral, and sketch it.

5. We can find a parametrization of the curve obtained by intersecting the cone $z = \sqrt{x^2 + y^2}$ and the plane $y = 1 + x$ as follows:

   (a) Parametrize the line $y = 1 + x$ in the xy-plane.

   (b) Now that you know how x and y depend on t, use $z = \sqrt{x^2 + y^2}$ to get a formula for z in terms of t.

   (c) In general, what do you need to know about the algebraic forms of the two surfaces in order to use an approach like this? In particular, try a similar approach for the curves obtained by intersecting the pairs of surfaces:

      i. $z = x^2 + y^2$ and $x + y = 2$.

      ii. $x^2 + 2y^2 + z^2 = 4$ and $x = z^2$.

      iii. $x^2 + y^2 = 4$ and $z = xy$ [**Hint**: This one is a bit different: Start by parametrizing the circle $x^2 + y^2 = 4$ in the xy-plane.]

      iv. $x^2 + 9y^2 = 36$ and $y + z = 1$.

   If you have extra time, try sketching these curves.