Math 2415

Problem Section #5

Make sure you do some problems from each section.

13.2 & 13.3: Calculus of Curves

1. This question asks you to review your understanding of the geometric meaning of the derivative of a parametrized curve.

   (a) Make a sketch of the curve $(x, y) = \mathbf{r}(t) = (t^2, t)$ for $0 \le t \le 2$.

   (b) Add the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.1)$, and $\mathbf{r}(1.1) - \mathbf{r}(1)$ to your sketch. [Don't worry about getting the points exactly correct!]

   (c) Calculate the tangent vector, $\mathbf{r}'(1)$, and add it to your sketch.

   (d) Calculate the secant vector
   $$\frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$$
   add it to your sketch, and explain why it is almost the same as $\mathbf{r}'(1)$.

2. Let C be the curve in $\mathbb{R}^2$ parametrized by $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}$, for $-2 \le t \le 2$.

   (a) Show that the curve C is given by $x^2 = y^3$.

   (b) Sketch the curve C. **Hint:** C has a cusp at the origin.

   (c) Calculate $\mathbf{r}'(0)$.

   (d) Suppose now that $\mathbf{r}$ is the position of a particle moving in the plane as a function of time. Describe the motion of the particle, especially near $t = 0$.

   (e) Calculate the speed and the acceleration vector of the particle at time $t = 1$.

3. Let C be the curve $(x, y, z) = \mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$ for $0 \le t \le 2$.

   (a) Parametrize the tangent line to C at $t = 1$ in such a way that the motion along the tangent line is a good approximation to the motion along the curve near $t = 1$.

   (b) Set up a one-dimensional integral for the length of C.

   (c) **[Trick]** Show that the function under the square root in the integrand is a perfect square, that is it can be expressed in the form $[g(t) + h(t)]^2$ for some functions g and h.

   (d) Hence, evaluate the integral.

   (e) If we changed the curve to $(x, y, z) = \mathbf{r}(t) = \sqrt{3}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$ could you still do the problem? Why? *Hint:* Nope!!

4. Let C be the curve $(x, y, z) = \mathbf{r}(t) = t^2\mathbf{i} + 9t\mathbf{j} + 4t^{3/2}\mathbf{k}$ for $0 \le t \le 1$. Use the same trick as in the previous problem to calculate the length of C.

14.1, Functions of Several Variable

1. Let $z = f(x, y) = e^{-x-y}$. Sketch the contours of $f(x, y) = k$ for $k = 0.5$, $k = 1$, $k = 2$. Use this information to help you sketch the graph of f.
2. Sketch a contour map for the function $z = f(x, y) = y^{1/3} + x$. *Hint:* Solve $y^{1/3} + x = k$ for y in terms of x.
3. 14.1.61
4. 14.1.62
5. You could also try 14.1.63–14.1.66 if you like!

Exam I Review

1. (From Fall 2010, Exam 1) Find the traces (i.e., slices) of the surface

$$x^2 \;=\; 1 + \frac{y^2}{4} + \frac{z^2}{9}$$

   in the planes $y = 0$, $z = 0$, and $x = k$, for $k = 0, \pm 1, \pm 2, \pm 3$. Then sketch the surface and name it.

2. (From Fall 2009, Exam 1)
   (a) Find a vector parametric equation for the line through the point $(1, 2, -1)$ that is normal to the plane $2x - y + 3z = 12$.
   (b) Find a parametrization of the plane containing the point $(1, -2, 1)$, $(2, -1, 0)$ and $(3, -2, 2)$.