Math 2415
Problem Section #7

Make sure you do some problems from each section.

14.6: Gradients and Directional Derivatives

1. Let $f(x, y) = e^y \cos(x)$ and let $\mathbf{x}_0 = (\pi/3, 0)$.

   (a) Find the gradient of f.

   (b) Evaluate the gradient of f at $\mathbf{x}_0$, i.e., find $\nabla f(\mathbf{x}_0)$.

   (c) Find the directional derivative of f at $\mathbf{x}_0$ in the direction $\theta = 3\pi/4$.

   (d) Find the directional derivative of f at $\mathbf{x}_0$ in the direction of the vector $\mathbf{v} = (3, 4)$.

   (e) Find the maximum rate of change of f at $\mathbf{x}_0$ and the direction in which it occurs.

   (f) In what direction is the minimum rate of change of f at $\mathbf{x}_0$?

   (g) Use the gradient of f to calculate the equation of the tangent line to the level curve $f(x, y) = \frac{1}{2}$.

14.7A: Local Optimization

A guided example

Consider the function $z = f(x, y) = x^2 + 3y^2 - 2xy - 3x$. The critical points of this function are the points in the xy-plane that satisfy the equations

$$0 = \frac{\partial f}{\partial x} = 2x - 2y - 3 \tag{1}$$

$$0 = \frac{\partial f}{\partial y} = 6y - 2x. \tag{2}$$

Equations (1) and (2) are the equations of a pair of lines in the xy-plane. The critical points are the points that lie on **both** of these lines (since both equations need to hold at a critical point). By sketching the two lines you can see that in this example there can only be one critical point of f.

In general the critical point equations

$$0 = \frac{\partial f}{\partial x}(x, y) \tag{3}$$

$$0 = \frac{\partial f}{\partial y}(x, y) \tag{4}$$

are the equations of a **pair of curves** in the xy-plane. The critical points are the points that lie on both curves. Often you can sketch these two curves and use your sketch to determine how many critical points there are and roughly where they are located. Then you can use this geometric information to guide an algebraic calculation to solve Equations (3) and (4) and thereby find the precise location of the critical points. **Use this combined geometric-algebraic technique in the problems below.**

Problems

1. Find the local maxima, minima, and saddle points of the following functions

   (a) $f(x, y) = x^4 + y^2 + 2xy$
   (b) $f(x, y) = y^3 - 3y + 3x^2y$
   (c) $f(x, y) = x^4 - 2x^2 + y^3 - 6y$
   (d) $f(x, y) = xy - e^{xy}$

14.3-14.5: Partial Derivatives, Tangent Planes, and Chain Rule

1. Let $f(x, y) = xy + x^2 = x(y + x)$ Calculate f_x, f_y, f_{xy} and f_{xx} at the point $\mathbf{x}_0 = (1, 1)$. Discuss the geometric meaning of each of these derivatives, with the aid of schematic diagrams showing slices of f in planes where x or y are constant.

2. Show that the function $u(x, y) = e^{-x}\cos(y)$ satisfies Laplace's equation $u_{xx} + u_{yy} = 0$.

3. Show that the function $u(x, t) = \cos(kx)\sin(akt)$ satisfies the wave equation $u_{tt} = a^2u_{xx}$.

4. Find a level set equation of the tangent plane to the function $z = f(x, y) = e^x\cos(xy)$ at $(x_0, y_0) = (0, 0)$. Explain why your solution shows that $e^x\cos(xy) \approx x + 1$ near $(0, 0)$.

5. 14.5.7

6. 14.5.15

7. 14.5.35