Math 2415
Problem Section #2

Make sure you do some problems from each section.

12.4: Cross Products

For Questions 3-7 below you must draw a schematic diagram (sketch, picture) that shows the relationships between the various points, vectors, planes etc in the problem before attempting to solve it algebraically. Do not make these pictures entirely realistic. For example, if you have a point $p = (1, 2, 3)$ in a plane P don't draw it exactly. Just draw any old point in any old plane.

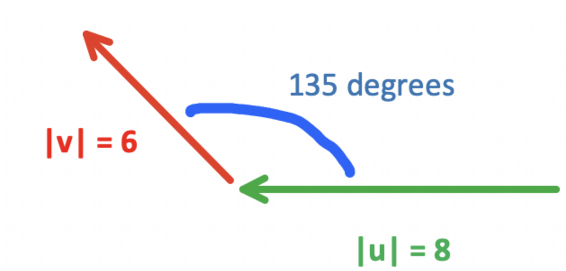
1. **[V]** Let $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$. Calculate $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

2. **[V]** Use properties of cross products (rather than the formula for the cross product in terms of a determinant) to calculate $(\mathbf{i} - 2\mathbf{k}) \times (3\mathbf{i} + 4\mathbf{j})$.

3. **[P]**

   Answer this problem using the picture below. You are *not* allowed to calculate the components of the vectors $\mathbf{u}$ and $\mathbf{v}$. *Warning:* Look carefully at the directions of the arrows on the vectors. Relate to theory from lectures!

   (a) Find $|\mathbf{v} \times \mathbf{u}|$
   (b) Determine whether $\mathbf{v} \times \mathbf{u}$ is directed in or out of the page
   (c) Use vector algebra to calculate the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.



4. **[V]** Find two unit vectors orthogonal to both $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (1, 0, -2)$.

5. **[P]** Let $A = (1, 2, 3)$, $B = (0, 1, -1)$, $C = (2, 4, -3)$, and $D = (1, 0, 2)$

   (a) Find a vector orthogonal to the plane that contains the points A, B, and C.
   (b) Find the area of the triangle ABC.
   (c) Find the volume of the parallelepiped three of whose edges are given by AB, AC, and AD.

6. **[P]**

   (a) Explain why there is at least one vector **v** so that $(1, 2, 4) \times \mathbf{v} = (2, -3, 1)$.

   (b) Is there a vector **v** so that $(1, 2, 4) \times \mathbf{v} = (2, 3, -1)$?

7. **[P]** Find all vectors **v** so that $\mathbf{i} \times \mathbf{v} = \mathbf{k}$. Argue both geometrically and algebraically.

8. [**Challenge**] Let **u** be a unit vector and let $\mathbf{w} \perp \mathbf{u}$.

   (a) Show that all the vectors **v** which have the property that $\mathbf{u} \times \mathbf{v} = \mathbf{w}$ are of the form $\mathbf{v} = c\mathbf{u} + \mathbf{w} \times \mathbf{u}$ for some scalar, $c \in \mathbb{R}$.

   (b) Explain why these vectors **v** are the position vectors of points that lie on a line, L.

   (c) Make a sketch showing the relationship between the vectors **u** and **w**, and the line, L.

   (d) Explain how the answer you obtained to Problem 7 is a special case of the formula in (a).

9. [**Challenge**] What must you know about a pair of nonzero vectors, **a** and **b**, to ensure that there is a vector, **x**, so that $\mathbf{x} \times \mathbf{a} = \mathbf{b}$ and $\mathbf{x} \cdot \mathbf{a} = |\mathbf{a}|$? In this situation how many such vectors **x** are there? Argue both geometrically and algebraically.

12.5A: Lines

Recall the following definitions:

(i) A **vector parametrization** of the line through the endpoint of the vector **a** in the direction of the vector **b** is given by $\mathbf{r}(t) = \mathbf{a} + t\mathbf{b}$, where $t \in \mathbf{R}$.

(ii) A **scalar parametrization** of the line in (i) is

$$
\begin{aligned}
x &= a_1 + tb_1 \\
y &= a_2 + tb_2 \\
z &= a_3 + tb_3
\end{aligned}
$$

where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$.

For each problem start by drawing a schematic diagram that illustrates the geometrical relationships between the various points, lines, and vectors in the problem. Use your diagram to help you set up equations that will help you solve the problem.

1. **[V]** Find vector and scalar parameterizations of the line through $P = (1, 2, -1)$ and $Q = (-1, 0, 2)$.

2. **[P]** Find *two* vector parameterizations of the line through the point $(3, 2, 1)$ that is parallel to the line $x = 2 + 3t$, $y = 4t$ $z = -1 + 5t$.

3. **[V]** Find a vector parameterizations of the line through the point $P = (1, 2, -1)$ that is perpendicular to both the vectors $\mathbf{i} - \mathbf{2}j$ and $\mathbf{j} + 3\mathbf{k}$.