Math 2415
Problem Section #9

Make sure you do some problems from each section.

14.8: The Method of Lagrange Multipliers, aka Constrained Optimization

Solution Strategy

For the problems from 14.8 you will use the method of Lagrange Multipliers to solve constrained optimization problems of the following type:

Find the absolute maximum and minimum of
$z = f(x, y)$ *subject to a constraint* $g(x, y) = k$.

In class we showed that if (x_0, y_0) is the location of such a maximum or minimum, then there is a scalar λ so that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \tag{1}$$
$$g(x_0, y_0) = k. \tag{2}$$

In other words, the triples (x_0, y_0, λ) satisfying these equations are the candidates for the solutions to our constrained optimization problem. So we call them critical points.

For the problems below, first solve the problem geometrically and then algebraically.

For the **geometric approach**, your task is to *visually spot those points,* (x_0, y_0)*, on the constraint curve which have the property that the tangent line to the constraint curve is equal to the tangent line to the level curve of* f *that goes through the point* (x_0, y_0). These points correspond to the critical points of the problem. (See question (1) below.) By solving the problem geometrically/pictorially, you can often quickly determine the number of critical points as well as their approximate locations. In some cases it may also enable you to find the exact locations of the critical points exactly.

For the **algebraic approach** you need to solve three equations in three unknowns, (x, y, λ) for the critical points as we discussed in class. You should use the geometric solution to guide/check your algebraic calculation, and in particular to make sure you have found all the critical points.

Problems

1. Suppose that (x_0, y_0) is a point on the constraint curve $g(x, y) = k$ so that the tangent line to the constraint curve agrees with the tangent line to the level curve of f that goes through (x_0, y_0). Make a sketch to explain why there is a scalar λ so that the triple (x_0, y_0, λ) satisfies equations (1) and (2).

2. Use the method of Lagrange multipliers to find the maximum and minimum of $z = f(x, y) = ye^x$ on the circle $x^2 + y^2 = 2$.

3. Suppose that (x_0, y_0) is a point on the constraint curve $g(x, y) = k$ so that $\nabla f(x_0, y_0) = (0, 0)$. Explain why (x_0, y_0) with $\lambda = 0$ solves equations (1) and (2). This tells us that if

   (a) (x_0, y_0) is a critical point of the unconstrained optimization problem (see Chapter 14.7)

   Find the maxima and minima of $z = f(x, y)$ *where* $(x, y) \in \mathbb{R}^2$

and

(b) (x_0, y_0) lies on the constraint curve,

then (x_0, y_0) with $\lambda = 0$ is a critical point for the constrained optimization problem Note: If you use the geometric method as in Question 1, you will not find critical points with $\lambda = 0$, so you need to check for them separately.

4. Use the method of Lagrange multipliers to find the maximum and minimum of $z = f(x, y) = xy^2$ on the circle $x^2 + y^2 = 1$. **Hint:** There are 6 critical points. Two of them can be found using the method in Question 3. You can find the approximate locations of the other four using the method in Question 1.

5. (a) Use the method of Lagrange multipliers to find the maximum and minimum of $z = f(x, y) = x^2 + y^2$ on the circle, C, given by $(x-1)^2 + y^2 = 1$.
   (b) Explain why the solutions to this problem gives the points on the circle, C, that are closest to and furthest from the origin.
   (c) There are two critical points for this Lagrange multipliers problem, one of which has $\lambda = 0$. Explain why the critical point with $\lambda = 0$ is also a critical point for the problem of finding the local maxima and minima of the function $z = f(x, y)$ in the entire plane, $\mathbb{R}^2$ (rather than on the circle, C).

6. In this problem you will find the absolute max and min of the function $z = f(x, y) = x^2 + y^2 - 4x - 4y$ on the closed disk $x^2 + y^2 \leq 9$.
   (a) Find critical points of f in the open disk $x^2 + y^2 < 9$.
   (b) Explain why the problem of finding the absolute maximum and minimum of the function $f(x, y) = x^2 + y^2 - 4x - 4y$ on the circle $x^2 + y^2 = 9$ is the same as the problem of finding the absolute maximum and minimum of the function $h(x, y) = 9 - 4x - 4y$ on the circle $x^2 + y^2 = 9$.
   (c) To solve the problem of optimizing the function $z = h(x, y)$ on the circle $x^2 + y^2 = 9$ you can just use the geometric approach outlined above. The symmetry of the problem will enable you to find the exact locations of the critical points!

15.1 Double Integrals over Rectangles

1. Evaluate the double integral, $\iint_R (2y + 3)\, dA$, where $R = [0, 3] \times [0, 2]$ by identifying it as the volume of a solid.

2. Calculate $\iint_R y e^{xy}\, dA$ where $R = [0, 1] \times [0, 2]$.

3. Sketch the solid enclosed by the paraboloid $z = 2 + x^2 + y^2$ and the planes $x = -1$, $x = +1$, $y = 0$, $y = 2$, and $z = 8$. Set up a double integral to calculate the volume of this solid and evaluate the integral.