Math 2415

Paper Homework #7

Chain Rule for Functions on Curves

Let $z = f(x, y)$ be a function on the plane and let $(x, y) = \mathbf{r}(t)$ be a curve in the plane. The composition

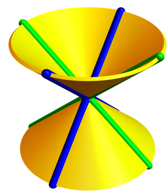
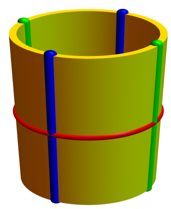
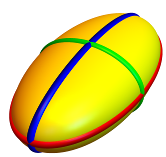
$$z = g(t) = f(\mathbf{r}(t)) = f(x(t), y(t))$$

is a scalar-valued function of one variable. The function g is called the **restriction** of f to the curve $\mathbf{r}$, since it just gives us the values of f along the curve $\mathbf{r}$. In this context, the Chain Rule for Functions on Curves states that

$$g'(t) \;=\; \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \;=\; f_x(x(t), y(t))x'(t) \;+\; f_y(x(t), y(t))y'(t).$$

1. **14.5, Chain Rule:** Let $z = f(x, y) = y^3 \sin x$ where $(x, y) = \mathbf{r}(t) = (\cos t, t^2)$.
   (a) Form the composition $g(t) = f(x(t), y(t))$ and then use the single variable chain rule to calculate $g'(t)$.
   (b) Use the Chain Rule for Functions on Curves to calculate $g'(t)$.
2. **14.5, Chain Rule:** Suppose that $z = f(x, y) = \sin(2x + 3y)$ where $x = x(t)$ and $y = y(t)$. If $x(0) = \pi/4$, $y(0) = \pi/3$, $x'(0) = 1$, and $y'(0) = -2$, find $\frac{dz}{dt}$ at $t = 0$.
3. **14.5, Chain Rule:** Let $z = f(x, y)$ where $x = r\cos\theta$ and $y = r\sin\theta$.
   (a) Show that
   $$\frac{\partial z}{\partial r} = f_x \cos\theta + f_y \sin\theta \tag{1}$$
   and
   $$\frac{1}{r}\frac{\partial z}{\partial \theta} = -f_x \sin\theta + f_y \cos\theta. \tag{2}$$
   (b) You can think of (1) and (2) as a system of two linear equations for unknowns f_x and f_y. Solve this system to find formulae for f_x and f_y in terms of $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$. **Hint:** Multiply (1) by $\sin\theta$ and (2) by $\cos\theta$ and follow your nose.
   (c) Show that $(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 = (\frac{\partial z}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial z}{\partial \theta})^2$.
4. **16.6, Parametrized Surfaces:**
   (a) Identify the surface with parametrization
   $$x = 3\cos\theta\,\sin\phi \qquad y = 4\sin\theta\,\sin\phi \qquad z = 5\cos\phi$$
   where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. **Hint:** Find an equation of the form $F(x, y, z) = 0$ for this surface by eliminating θ and ϕ from the equations above.
   (b) Calculate a parametrization for the tangent plane to the surface at $(\theta, \phi) = (\pi/3, \pi/4)$.

5. **16.6, Parametrized Surfaces:** In each of the following surface parametrizations, filling in the missing two entries. The highlighted curves are "grid curves" for the parametrization. Justify your work.

a)		The surface $\boxed{x^2 + y^2 = z^2}$ is parametrized by $\vec{r}(\theta, z) = \langle \ldots\ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots\ldots, z \rangle$
b)		The surface $\boxed{x^2 + y^2 = 1}$ is parametrized by $\vec{r}(\theta, z) = \langle \ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots, z \rangle$
c)		The surface $\boxed{2(x-1)^2 + (y-5)^2 + 4z^2 = 1}$ is parametrized by $\vec{r}(\theta, \phi) = \langle \ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots, \cos(\phi)/2 \rangle$

6. **16.6, Parametrized Surfaces:**Find a parametrization for each of the following surfaces. [Note: There are many correct answers!] Show how you arrived at your answer. **Hint:** It is often helpful to construct your parametrization using (i) cylindrical coordinates, (ii) spherical coordinates, or (iii) by using a parametrization such as $(x, y, z) = (u, v, f(u, v))$ for the surface $z = f(x, y)$. **Make sure you specify the range of values of the parameters.**

   (a) The portion of the sphere $x^2 + y^2 + z^2 = 3$ that is between the planes $z = -\sqrt{3}/2$ and $z = \sqrt{3}/2$.

   (b) The portion of the plane $x + y + z = 1$ inside the cylinder $x^2 + y^2 = 9$. **Hint:** Use (r, θ) as your parameters.