Math 2415

Problem Section #1

Make sure you do some problems from each section.

12.1: 3D Coordinate Systems

1. Draw a rectangular box with the origin and the point $(1, 2, 3)$ as opposite vertices and faces parallel to the coordinate planes. Label each vertex with its coordinates. Find the length of the diagonal of the box.

2. (a) What does the equation $x = 2$ represent in $\mathbb{R}^2$? Sketch!
   (b) What does the equation $x = 2$ represent in $\mathbb{R}^3$? Sketch!
   (c) What does the equation $z = 1$ represent in $\mathbb{R}^3$? Sketch!
   (d) Describe the set of all points, (x, y, z), in $\mathbb{R}^3$ for which $x = 2$ and $z = 1$. Sketch!

3. For what values of b and c do the points $(1, 2, 3)$, $(4, 5, 1)$, and $(10, b, c)$ all lie on the same line?

4. (a) Find the equation of the sphere with center $(1, 3, 5)$ and radius 4.
   (b) What is the intersection of this sphere with the xz-plane? Argue algebraically and geometrically.
   (c) What would the radius of the sphere have to be for the the intersection of the sphere and the xz-plane to be a single point. What are the coordinates of this point?

12.2: Vectors

1. Do not use coordinate representations of vectors to solve this problem. Just draw pictures.
   (a) Draw two vectors that are not parallel and label them $\mathbf{a}$ and $\mathbf{b}$.
   (b) Sketch the vector $\mathbf{a} + \mathbf{b}$
   (c) Sketch the vector $\mathbf{a} - \frac{1}{2}\mathbf{b}$
   (d) Sketch the vector $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

2. Sketch a parallelogram and label the vertices A, B, C, and D going around counter-clockwise from the bottom left vertex. Let E be the point obtained by intersecting the two diagonals of the parallelogram. Make sure the side lengths of your parallelogram are not all equal, ie you did not draw a rhombus. The notation $\overrightarrow{AB}$ refers to the displacement vector from the point A to the point B.
   (a) Name all pairs of equal vectors in your sketch.
   (b) Write each combination of vectors as a single vector: $\overrightarrow{AB} + \overrightarrow{BC}$, $\overrightarrow{AE} - \overrightarrow{EB}$, $2\overrightarrow{AB} + \overrightarrow{BD}$.

3. Let $\mathbf{a} = 3\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Find
   (a) $\mathbf{a} + 2\mathbf{b}$
   (b) $|\mathbf{b}|$
   (c) $|\mathbf{a} - \mathbf{b}|$.

4. Suppose that $\mathbf{v} \in \mathbb{R}^2$ lies in the 2nd quadrant, makes an angle of $120°$ with the positive x-axis, and has length $|\mathbf{v}| = 2$. Find the coordinates of $\mathbf{v}$.

12.3: The Dot Product

1. Find $\mathbf{a} \cdot \mathbf{b}$ if

   (a) $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (-2, 3)$,

   (b) $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$,

   (c) $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$, and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $120°$.

2. (a) Let $\mathbf{u} = (3, -2, 1)$ and $\mathbf{v} = (2, 4, -1)$.

   (b) Find the scalar and vector projections of $\mathbf{u}$ onto $\mathbf{v}$.

   (c) Find the angle between $\mathbf{u}$ and $\mathbf{v}$ to the nearest degree (use a calculator!)

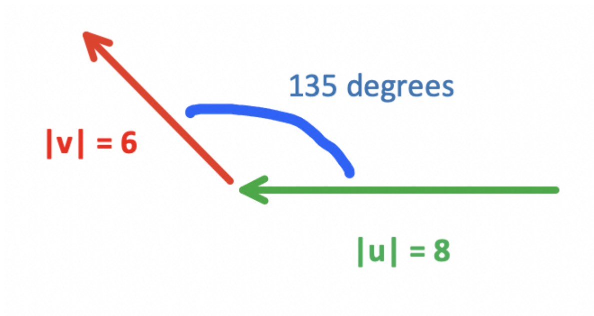
   (d) Find three nonzero vectors that are orthogonal to $\mathbf{u}$.

3. Answer this problem using the picture below. You are *not* allowed to calculate the components of the vectors $\mathbf{u}$ and $\mathbf{v}$. *Warning:* Look carefully at the directions of the arrows on the vectors. Relate to theory from lectures!

   (a) Find $\mathbf{u} \cdot \mathbf{v}$

   (b) Use triangle geometry to find the scalar projection of $\mathbf{v}$ onto $\mathbf{u}$.

   (c) Use triangle geometry to find the vector projection of $\mathbf{u}$ onto $\mathbf{v}$. (Write your answer in terms of $\mathbf{v}$.)



4. Let C be the point on the line segment AB that is twice as far from A and it is from B, and let O denote the origin. Let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, and $\mathbf{c} = \overrightarrow{OC}$. [Recall that $\overrightarrow{AB}$ refers to the displacement vector from the point A to the point B.]

   (a) Make a sketch showing the relationships between all these points and vectors. Your sketch will help you solve the other parts of the problem.

   (b) Express $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

   (c) Hence express $\overrightarrow{AC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

   (d) Hence find a formula for $\mathbf{c}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

   (e) Now also suppose that $\mathbf{a} \perp \mathbf{b}$ and $|\mathbf{b}| = 1$. Find the scalar projection of $\mathbf{c}$ onto $\mathbf{b}$. *Hint:* Your answer should be a number.

   (f) Calculate $\mathbf{c}$ in the special case that $\mathbf{a} = \mathbf{i}$ and $\mathbf{b} = \mathbf{j}$. Is your answer consistent with your answer to (e)?

Extra Challenge Questions:

1. Find the angle between a diagonal of a cube and a diagonal of one of its faces.
2. We learned in class that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$. Use this property to prove the Cauchy-Schwarz inequality
$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|. \tag{1}$$
3. The triangle inequality states that
$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|. \tag{2}$$
   (a) Give a geometric interpretation of the triangle inequality.
   (b) Use the fact that $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$ and the distribute law for the dot product to prove (2).
4. The parallelogram law states that
$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2. \tag{3}$$
   (a) Give a geometric interpretation of the parallelogram law.
   (b) Prove the parallelogram law.
5. Let $\mathbf{a} = 2\mathbf{i} + 4\mathbf{k}$.
   (a) Find one vector $\mathbf{b}$ so that $\text{Comp}_{\text{a}}(\mathbf{b}) = 2$.
   (b) Draw a picture to convince yourself that there are an infinite number of vectors, $\mathbf{b}$ for which $\text{Comp}_{\text{a}}(\mathbf{b}) = 2$. Describe this set of vectors using an English sentence.