Find a parametrization of the following:

$\boxed{1}$ $x^2+y^2=9$ $\quad \vec{r}(t) = \langle 3\cos t, 3\sin t \rangle, \quad 0 \le t \le 2\pi$

$\boxed{2}$ $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ $\quad \vec{r}(t) = \langle 2\cos t, 3\sin t \rangle, \quad 0 \le t \le 2\pi$

$\boxed{3}$ line segment from $(1,2)$ to $(3,5)$

$$\vec{r}(t) = \langle 1,2 \rangle + t\left(\langle 3,5 \rangle - \langle 1,2 \rangle\right)$$

$$= \langle 1,2 \rangle + t\langle 2,3 \rangle = \langle 1+2t, 2+3t \rangle$$

$$0 \le t \le 1$$

$\boxed{4}$ $x^2+y^2+z^2=16$

$$\vec{r}(\theta,\phi) = \langle 4\sin\phi\cos\theta, 4\sin\phi\sin\theta, 4\cos\phi \rangle;$$

$$0 \le \phi \le \pi$$
$$0 \le \theta \le 2\pi$$

$\boxed{5}$ $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{25} = 1$

$$\vec{r}(\theta,\phi) = \langle 2\sin\phi\cos\theta, 3\sin\phi\sin\theta, 5\cos\phi \rangle$$

$$0 \le \phi \le \pi ; \quad 0 \le \theta \le 2\pi$$

$\boxed{6}$ $(x-5)^2 + (y+2)^2 = 16$

$$\vec{r}(t) = \langle 5+4\cos t, -2+4\sin t \rangle ; \quad 0 \le t \le 2\pi$$

$\boxed{12.5}$ — Chain Rule

$\boxed{1}$ Let C be a curve in xy plane with parametrization $\vec{r}(t)$. Given that $f(x,y) = e^x - y^2$, $\vec{r}(2) = \langle 0, 1\rangle$ and $\vec{r}\,'(2) = \langle -2, 3\rangle$.

Let $z = f(\vec{r}(t))$. Find $\frac{dz}{dt}$ at $t=2$

Solution: Chain Rule yields :

$$\frac{dz}{dt} = \frac{d}{dt}\left(f(\vec{r}(t))\right) = \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$\Rightarrow \left.\frac{dz}{dt}\right|_{t=2} = \vec{\nabla} f(\vec{r}(2)) \cdot \vec{r}\,'(2)$$

$$= \vec{\nabla} f(0,1) \cdot \langle -2, 3\rangle$$

$$= \langle 1, -2\rangle \cdot \langle -2, 3\rangle$$

$$= -2 - 6 = \boxed{-8}$$

$$\vec{\nabla} f = \langle f_x, f_y\rangle$$
$$= \langle e^x, -2y\rangle$$

$$\Rightarrow \vec{\nabla} f(0,1)$$
$$= \langle 1, -2\rangle$$

Recall from Calc I :

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)).g'(x)$$

$f \rightarrow$ function of several var

$$\frac{d}{dt}\left(f(\vec{r}(t))\right)$$
$$= \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

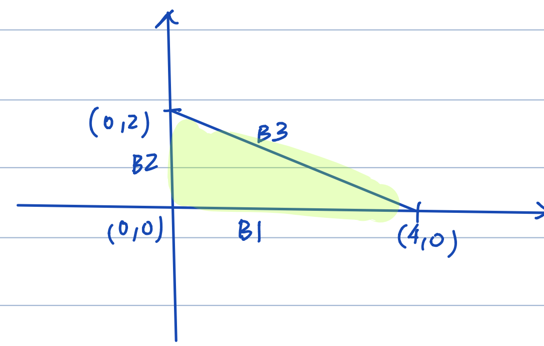
↑ dot prod

$\boxed{14.7}$ Absolute Extrema

Find the absolute maximum and minimum values of f on the set D.

$f(x,y) = x + y - xy$; D is the closed triangular region bounded by $(0,0)$, $(0,2)$ and $(4,0)$

Solution :



Interior: Find the CPs.

$f_x = 1 - y$

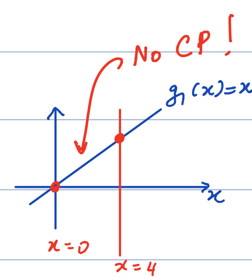
$f_y = 1 - x$

$f_x = 0 = f_y \Rightarrow x = y = 1$

$\Rightarrow$ CP : $A(1,1)$

$\boxed{f(1,1) = 1}$

B1 : $0 \le x \le 4$; $y = 0$



$\Rightarrow f(x,y) = f(x,0) = x$

f values at end points :

$$\boxed{\begin{array}{l} f(0,0) = 0 \\ f(4,0) = 4 \end{array}}$$

B2 : $0 \le y \le 2$; $x = 0$

$\Rightarrow f(x,y) = f(0,y) = y$

f values at end points:

$f(0,0) = 0$ (already taken)

$\boxed{f(0,2) = 2}$

B3 : $x + 2y = 4$ OR $x = 4-2y$; $0 \le y \le 2$

$$\Rightarrow f(x, y) = f(4-2y, y)$$
$$= (4-2y) + y - (4-2y)y$$
$$= 4 - y - 4y + 2y^2$$
$$= 4 - 5y + 2y^2 \; ; \quad 0 \le y \le 2$$
$$= g(y) \quad (\text{say})$$

$g'(y) = -5 + 4y = 0 \Rightarrow y = \frac{5}{4} \Rightarrow x = 4 - 2y = 4 - \frac{5}{2} = \frac{3}{2}$

CP : $\left(\frac{3}{2}, \frac{5}{4}\right)$

$$\boxed{\begin{aligned} f\left(\tfrac{3}{2}, \tfrac{5}{4}\right) &= \tfrac{3}{2} + \tfrac{5}{4} - \tfrac{15}{8} \\ &= \tfrac{7}{8} \end{aligned}}$$

end points :

$f(0, 2) = 2$

$f(4, 0) = 4$

Comparing f values in boxes, we get:

$f(4, 0) = 4$ (Max value)

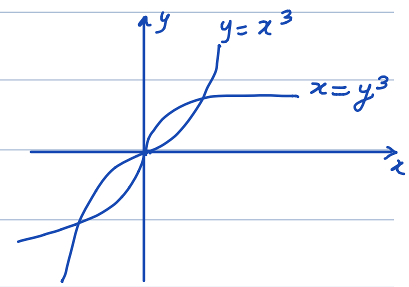
$f(0, 0) = 0$ (Min value)

$\boxed{14.7}$ Local Extrema

$\boxed{A}$ Find the local maximum, minimum and saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$.

Solution: $f_x = 4x^3 - 4y$ and $f_y = 4y^3 - 4x$

$f_x = f_y = 0 \Rightarrow y = x^3$ and $x = y^3$.



geometrically:
critical points: $(0,0)$, $(-1,-1)$ and $(1,1)$

algebraic method (to find CPs)

$y = x^3$ and $x = y^3$

$\Rightarrow y = (y^3)^3$

$\Rightarrow y^9 - y = 0$

$\Rightarrow y(y^8 - 1) = 0$

$\Rightarrow y =$	1	-1	0
$\Rightarrow x =$	1	-1	0

as $x = y^3$

Critical Points:

$(0,0)$, $(1,1)$, $(-1,-1)$

$f_{xx} = 12x^2$; $f_{xy} = f_{yx} = -4$; $f_{yy} = 12y^2$

	$D = f_{xx} f_{yy} - f_{xy}^2 = 144x^2y^2 - 16$	f_{xx}	Classification
$(0,0)$	$-$	$-$	Saddle pt @ $(0,0,1)$
$(-1,-1)$	$+$	$+$	Rel min @ $(-1,-1,-1)$
$(1,1)$	$+$	$+$	Rel min @ $(1,1,-1)$

value

value

$\boxed{B}$ $f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 60.$

$f_x = 6xy - 12x$ and $f_y = 3y^2 + 3x^2 - 12y$

$f_x = 0$ and $f_y = 0$

$\Rightarrow$ $6xy - 12x = 0$ and $3y^2 + 3x^2 - 12y = 0$

$\Rightarrow$ $6x(y-2) = 0$ and $y^2 + x^2 - 4y = 0$

$\Rightarrow$ $(x = 0$ OR $y = 2)$ and $x^2 + y^2 - 4y = 0$

$x = 0$ OR $y = 2$

and $x^2 + y^2 - 4y = 0$ | and $x^2 + y^2 - 4y = 0$

$\Rightarrow$ $y^2 - 4y = 0$ | $\Rightarrow$ $x^2 + 4 - 8 = 0$

$\Rightarrow$ $y = 0/4$ | $\Rightarrow$ $x = \pm 2$

$(0,0)$ $(0,4)$ | $(2,2)$, $(-2,2)$

$f_{xx} = 6y - 12$; $f_{yy} = 6y - 12$; $f_{xy} = 6x$

$\Rightarrow$ $D = (6y-12)^2 - (6x)^2.$

Critical Points	D	f_{xx} $\parallel$ $(6y-12)$	Class.
$(0,0)$	+	− ∩	Rel max.
$(0,4)$	+	+ ∪	Rel min
$(2,2)$	−	N/A	Saddle
$(-2,2)$	−	N/A	Saddle

$\boxed{15.1}$

Evaluate the integral $\displaystyle\iint_R y\, e^{-xy}\, dA$; $R = [0,2] \times [0,3]$

Solution:

$$\int_0^3 \int_0^2 y\, e^{-xy}\, dx\, dy$$

← Integrating with respect to x first seems easier!

$$= \int_0^3 y \left. \frac{e^{-xy}}{(-y)} \right|_{x=0}^{2} dy$$

$$= \int_0^3 1 - e^{-2y}\, dy = \left. y - \frac{e^{-2y}}{-2} \right|_0^3$$

$$= 3 + \frac{e^{-6} - 1}{2} = \boxed{\frac{e^{-6} + 5}{2}}$$

u-substitution yields:

$$\int e^{ax}\, dx = \frac{e^{ax}}{a} + C \quad \text{if } a \neq 0$$

$$\int_e^f e^{ax+by+c}\, dx = \left. \frac{e^{ax+by+c}}{a} \right|_{x=e}^{f} \quad \text{if } a \neq 0$$

$$\int_e^f e^{ax+by+c}\, dy = \left. \frac{e^{ax+by+c}}{b} \right|_{y=e}^{f} \quad \text{if } b \neq 0$$

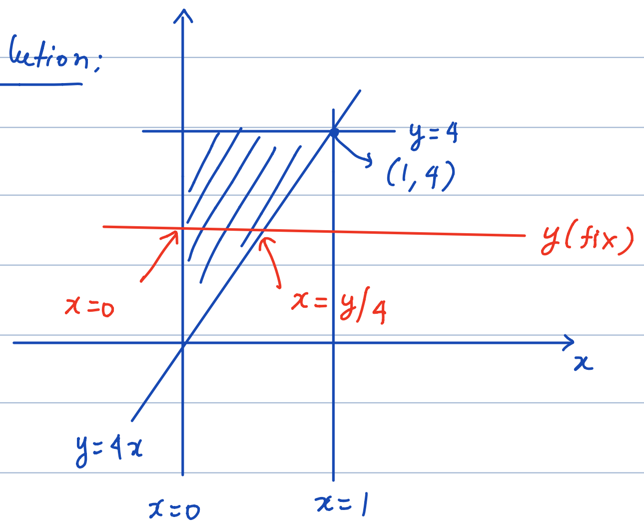
$\boxed{15.2}$

Evaluate the double integral $I = \int_0^1 \int_{4x}^4 e^{y^2}\,dy\,dx$

Solution:

$0 \le x \le 1$

and $4x \le y \le 4$



Changing the order of integration, we get,

$$I = \int_0^4 \int_0^{y/4} e^{y^2}\,dx\,dy = \int_0^4 e^{y^2} x \Big|_{x=0}^{y/4} dy$$

$$= \frac{1}{4}\int_0^4 \underbrace{e^{y^2}}_{e^u} \underbrace{y\,dy}_{\frac{du}{2}} = \frac{1}{8}\, e^{y^2}\Big|_{y=0}^{4} = \boxed{\frac{e^{16}-1}{8}}$$

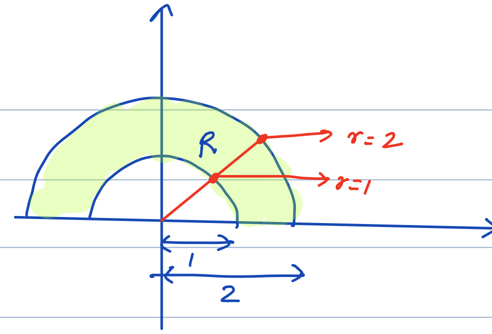
$u = y^2$

$\Rightarrow du = 2y\,dy$

$\Rightarrow \dfrac{du}{2} = y\,dy$

$\boxed{15.3}$

EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



$r=1$

$$I = \int_0^{\pi}\int_1^{2} (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi}\int_1^{2} 3r^2 \cos\theta \; dr\, d\theta + \int_0^{\pi}\int_1^{2} 4r^3 \sin^2\theta \; dr\, d\theta$$

$$= 3\left(\int_0^{\pi} \cos\theta\, d\theta\right)\left(\int_1^{2} r^2\, dr\right) + 4\left(\int_0^{\pi} \underbrace{\sin^2\theta}_{=\frac{1-\cos 2\theta}{2}}\, d\theta\right)\left(\int_1^{2} r^3\, dr\right)$$

$$= 3\left(\sin\theta\Big|_0^{\pi}\right)\left(\frac{r^3}{3}\Big|_1^{2}\right) + \frac{4}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi}\left(\frac{r^4}{4}\right)_1^{2}$$

$$= 0 + 2(\pi)\left(\frac{2^4 - 1^4}{4}\right)$$

$$= 2\pi\left(\frac{15}{4}\right) = \boxed{\frac{15\pi}{2}}$$

EXAMPLE 2 Evaluate $\iiint_E z\,dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

$$\iint_D \int_0^{1-x-y} z \, dz \, dA$$

$$= \iint_D \frac{z^2}{2} \Big|_0^{1-x-y} dA$$

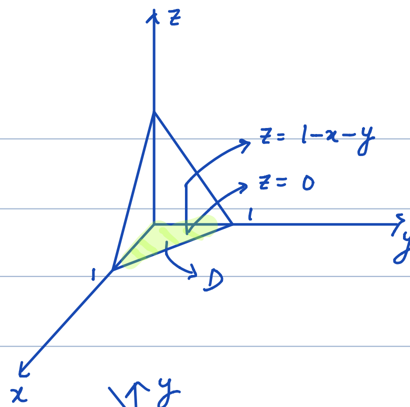
$$= \frac{1}{2} \iint_D (1-x-y)^2 \, dA$$

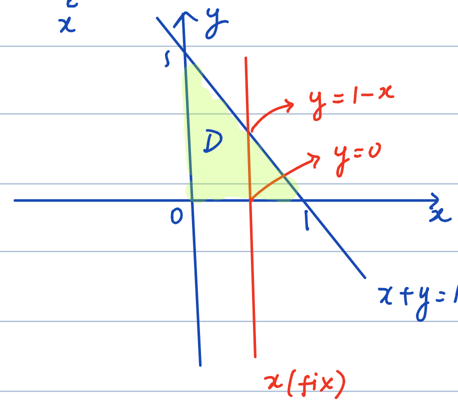
$$= \frac{1}{2} \int_0^1 \int_0^{1-x} (1-x-y)^2 \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \frac{(1-x-\boxed{y})^3}{3(-1)} \Big|_{y=0}^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 \frac{0-(1-x)^3}{-3} dx = \frac{1}{6} \int_0^1 (1-x)^3 dx$$

$$= \frac{1}{6} \frac{(1-x)^4}{-4} \Big|_0^1 = \boxed{\frac{1}{24}}$$





Note : u-sub gives:

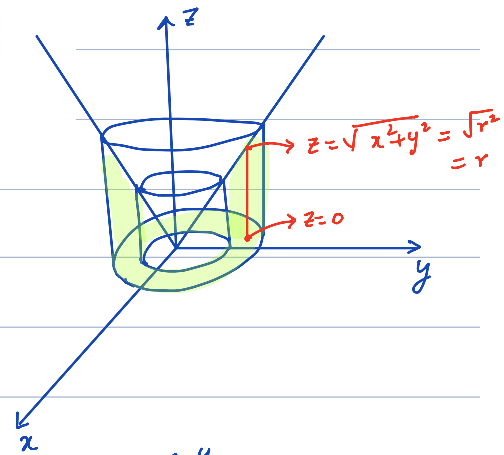
if $a \neq 0$, then $\int_0^1 (ax+by+c)^3 \, dx = \frac{(ax+by+c)^4}{4(a)} \Big|_{x=0}^{x=1}$ ← u-sub

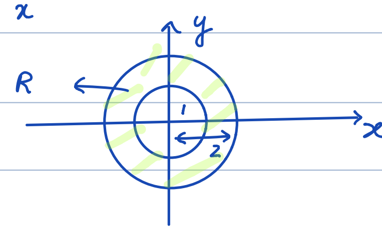
if $b \neq 0$, then $\int_0^1 (ax+by+c)^3 \, dy = \frac{(ax+by+c)^4}{4(b)} \Big|_{y=0}^{y=1}$

15.7

Find the volume of the solid

20. Below the cone $z = \sqrt{x^2 + y^2}$ and above the ring $1 \leqslant x^2 + y^2 \leqslant 4$





$$V = \iiint dV$$

$$= \iint_R \int_0^{\sqrt{x^2+y^2}} dz \quad dA$$

$$= \int_0^{2\pi} \int_1^2 \int_0^r dz \,(r\, dr\, d\theta)$$

$$= \int_0^{2\pi} \int_1^2 \left(z \Big|_0^r \right) r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_1^2 r^2\, dr\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta \right) \left(\int_1^2 r^2\, dr \right)$$

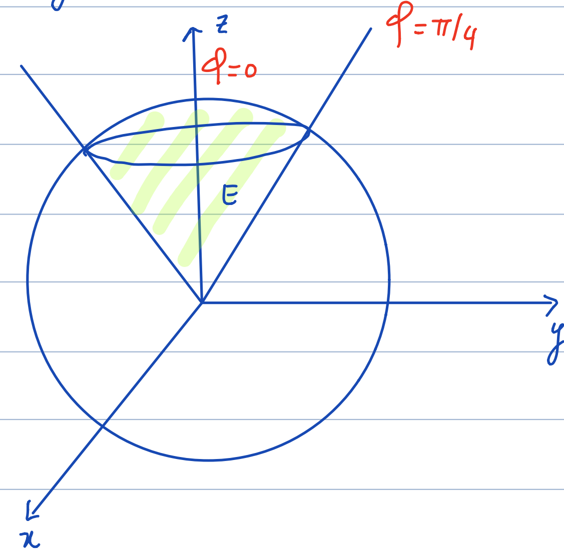
$$= \left(\theta \Big|_0^{2\pi} \right) \left(\frac{r^3}{3} \Big|_1^2 \right)$$

$$= (2\pi) \left(\frac{2^3 - 1^3}{3} \right)$$

$$= 2\pi \left(\frac{7}{3} \right) = \boxed{\frac{14\pi}{3} \text{ cubic units}}$$

$\boxed{15.8}$

Use spherical coordinates to find the volume of the solid above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=1$



$$V = \iiint_E dV = \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^1 \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \left(\int_0^{\pi/4} \sin\varphi \, d\varphi\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 \rho^2 \, d\rho\right)$$

$$= \left(-\cos\varphi \Big|_0^{\pi/4}\right)\left(\theta \Big|_0^{2\pi}\right)\left(\frac{\rho^3}{3}\Big|_0^1\right)$$

$$= \left(1 - \frac{\sqrt{2}}{2}\right)\left(2\pi\right)\left(\frac{1}{3}\right) = \frac{2\pi}{3}\left(1 - \frac{\sqrt{2}}{2}\right)$$

$\boxed{15.9}$

23. $\displaystyle\iint_R \frac{x-2y}{3x-y}\,dA$, where R is the parallelogram enclosed by the lines $x-2y=0$ $\boxed{u=0}$, $x-2y=4$ $\boxed{u=4}$, $3x-y=1$ $\boxed{v=1}$, and $3x-y=8$ $\boxed{v=8}$

Solution: Let $u = x-2y$ and $v = 3x-y$

$$\begin{aligned} 3x-6y &= 3u \\ -\ 3x-y &= v \end{aligned}$$

$\Rightarrow -5y = 3u - v$

$\Rightarrow \boxed{y = \dfrac{v-3u}{5}}$

$$\begin{aligned} 6x-\cancel{2y} &= 2v \\ -\ x-\cancel{2y} &= u \end{aligned}$$

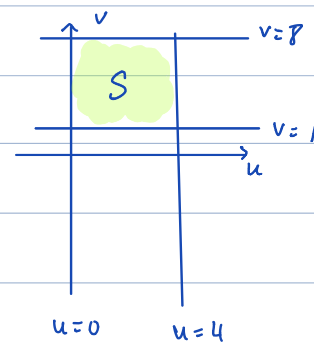
$\Rightarrow 5x = 2v-u$

$\Rightarrow \boxed{x = \dfrac{2v-u}{5}}$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{vmatrix}$$

$$= -\frac{1}{25} + \frac{6}{25} = \frac{1}{5} \Rightarrow |J| = \frac{1}{5}$$

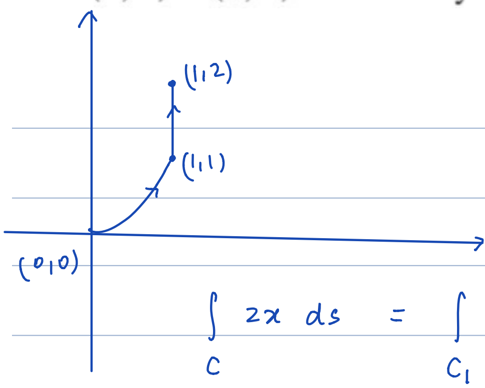


$$\Rightarrow \iint_R \frac{x-2y}{3x-y}\,dA = \iint_S \frac{u}{v}\,|J|\,du\,dv$$

$$= \frac{1}{5}\int_1^8\int_0^4 \frac{u}{v}\,du\,dv = \frac{1}{5}\left(\int_1^8 \frac{1}{v}\,dv\right)\left(\int_0^4 u\,du\right)$$

$$= \frac{1}{5}\left(\ln|v|\Big|_1^8\right)\left(\frac{u^2}{2}\Big|_0^4\right) = \boxed{\frac{8\ln 8}{5}}$$

EXAMPLE 2 Evaluate $\int_C 2x\, ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from (0, 0) to (1, 1) followed by the vertical line segment C_2 from (1, 1) to (1, 2).



$C_1 : \vec{r}(t) = \langle t, t^2 \rangle \; ; \quad 0 \le t \le 1$

$C_2 : \vec{r}(t) = \langle 1, 1 \rangle + t \langle 0, 1 \rangle$

$= \langle 1, 1+t \rangle \quad 0 \le t \le 1$

$$\int_C 2x\, ds = \underbrace{\int_{C_1} 2x\, ds}_{\mathrm{I}} + \underbrace{\int_{C_2} 2x\, ds}_{\mathrm{II}}$$

$\underline{\underline{\mathrm{I}}}$: $ds = |\vec{r}\,'(t)|\, dt = |\langle 1, 2t \rangle|\, dt = \sqrt{1+4t^2}\, dt$

$$\int_{C_1} 2x\, ds = \int_0^1 2(t)\sqrt{1+4t^2}\, dt = 2\int_0^1 \underbrace{\sqrt{1+4t^2}}_{u^{\frac{1}{2}}}\, \underbrace{t\, dt}_{\frac{du}{8}}$$

$$= \frac{1}{\cancel{8}_4}\cancel{2}\, \frac{(1+4t^2)^{3/2}}{3/2}\Bigg|_{t=0}^{1} = \frac{1}{6}\left(5^{3/2} - 1\right)$$

$\boxed{u = 1+4t^2 \quad du = 8t\, dt}$

$\underline{\underline{\mathrm{II}}}$: $ds = |\vec{r}\,'(t)|\, dt = |\langle 0, 1 \rangle|\, dt = \sqrt{0^2+1^2}\, dt = dt$

$$\int_{C_2} 2x\, ds = \int_0^1 2(1)\, dt = 2t\Big|_0^1 = 2$$

$$\Rightarrow \int_C 2x\, ds = \frac{1}{6}\left(5^{3/2} - 1\right) + 2 = \boxed{\frac{5^{3/2} + 11}{6}}$$

EXAMPLE 4 Evaluate $\int_C y^2\,dx + x\,dy$, where (a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$ and (b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$. (See Figure 7.)

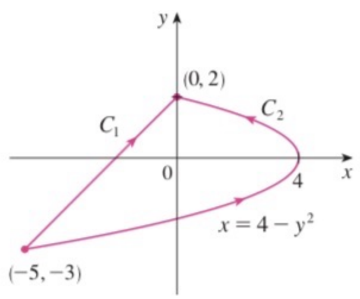


FIGURE 7

Parametrize C_1 & C_2:

$C_1 : \vec{r}(t) = \langle -5, -3 \rangle + t\langle 5, 5 \rangle$

$= \langle -5+5t, -3+5t \rangle \;;\; 0 \le t \le 1$

$C_2 : \vec{r}(t) = \langle 4 - t^2, t \rangle \;;\; -3 \le t \le 2$

$$\int_{C_1} y^2\,dx + x\,dy = \int_0^1 (-3+5t)^2(5\,dt) + (-5+5t)(5\,dt)$$

$$= \left[\frac{\cancel{5}(-3+5t)^3}{3(\cancel{5})} \Bigg|_{t=0}^{1} + \frac{\cancel{5}(-5+5t)^2}{\cancel{5}\,(2)} \Bigg|_{t=0}^{1} \right]$$

$$= \frac{1}{3}\left(2^3 + 3^3\right) + \frac{0^2 - 5^2}{2} = \frac{8+27}{3} - \frac{25}{2} = \boxed{-\frac{5}{6}}$$

$$\int_{C_2} y^2\,dx + x\,dy = \int_{-3}^{2} t^2(-2t\,dt) + (4-t^2)\,dt$$

$$= \int_{-3}^{2} -2t^3 + 4 - t^2\,dt = \frac{-2t^4}{4} + 4t - \frac{t^3}{3} \Bigg|_{-3}^{2}$$

$$= -\frac{1}{2}(16 - 81) + 4(5) - \left(\frac{8+27}{3}\right)$$

$$= \frac{65}{2} + 20 - \frac{35}{3} = \boxed{\frac{245}{6}}$$

integral is path dependent

$\boxed{16.3 - 16.4}$

$\boxed{*}$ Compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle y, -x \rangle$ and

C is the unit circle oriented counter-clockwise

Use the answer to determine whether $\vec{F}$ is conservative or not.

Solution: $C : \vec{r}(t) = \langle \cos t, \sin t \rangle \; ; \; 0 \le t \le 2\pi$

$\Rightarrow d\vec{r} = \langle -\sin t, \cos t \rangle \, dt$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle \sin t, -\cos t \rangle \cdot \langle -\sin t, \cos t \rangle dt$$

$$= \int_0^{2\pi} -\sin^2 t - \cos^2 t \; dt = \int_0^{2\pi} (-1)\, dt = -t \Big|_0^{2\pi}$$

$$= -2\pi$$

Since C is a closed curve and $\int_C \vec{F} \cdot d\vec{r} \neq 0$, therfore $\vec{F}$ is not conservative.

$\boxed{*}$ Let $\vec{F}(x,y) = (3 + 2xy)\hat{i} + (x^2 - 3y^2)\hat{j}$. Is the vector field $\vec{F}$ conservative?

Solution: $P = 3 + 2xy$ and $Q = x^2 - 3y^2$

$\Rightarrow P_y = 2x$ and $Q_x = 2x$

Since $P_y = Q_x$, therefore $\vec{F}$ is conservative.

$\boxed{16.3}$

EXAMPLE 4

(a) If $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by

$$\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j} \qquad 0 \leqslant t \leqslant \pi$$

(a) $F = \vec{\nabla} f \Rightarrow \langle 3 + 2xy,\ x^2 - 3y^2 \rangle = \langle f_x, f_y \rangle$

$\boxed{1}$ $f_x = 3 + 2xy \Rightarrow f = \int 3 + 2xy \, dx$

$= 3x + x^2 y + g_1(y)$

$\boxed{2}$ $f_y = x^2 - 3y^2$

$\Rightarrow f = \int x^2 - 3y^2 \, dy = x^2 y - y^3 + g_2(x)$

Comparing $\boxed{1}$ and $\boxed{2}$,

$f = x^2 y + 3x - y^3 + C$ and $\vec{F} = \vec{\nabla} f$

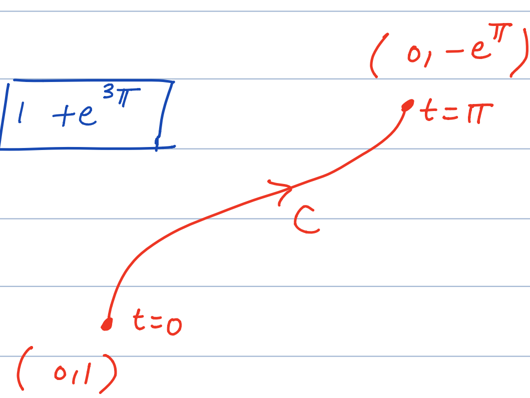
$\vec{F}$ is conservative

$\Rightarrow \int_C \vec{F} \cdot d\vec{r}$ is path independent.

(b) $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

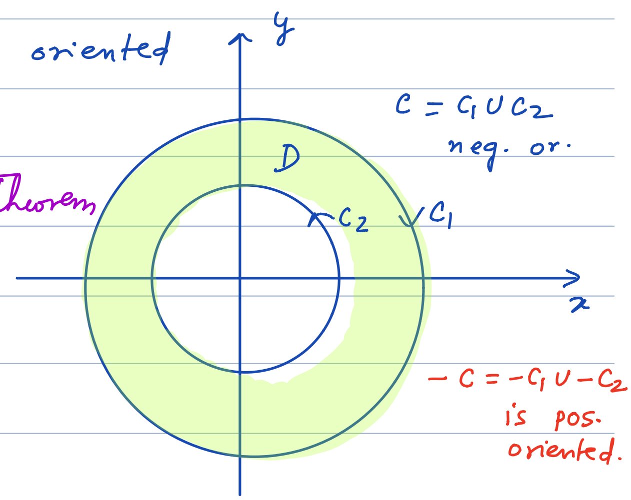
$= f(0, -e^{\pi}) - f(0, 1)$

$= -(-e^{\pi})^3 + 1 = \boxed{1 + e^{3\pi}}$



Use Green's theorem to evaluate the line integral $\int_C \underbrace{(1-y^3)}_{P}\,dx + \underbrace{(x^3+e^{y^2})}_{Q}\,dy$, where C is the boundary of the annulus, $4 \le x^2+y^2 \le 9$. You should orient C so that the inner circle is traversed counterclockwise and the outer circle is traversed clockwise.

Solution: C is negatively oriented



$$\Rightarrow \int_C \vec{F}\cdot d\vec{r}$$

apply Green's Theorem

$$= -\int_{\boxed{-C}} \vec{F}\cdot d\vec{r}$$

$-C$ → positively oriented.

$$= -\iint_D \Big(\overbrace{\frac{\partial Q}{\partial x}}^{3x^2} - \overbrace{\frac{\partial P}{\partial y}}^{-3y^2}\Big)\,dA \quad \text{(using Green's theorem)}$$

$$= -\iint_D 3(x^2+y^2)\,dA = -3\int_0^{2\pi}\int_2^3 r^2\,(r\,dr\,d\theta)$$

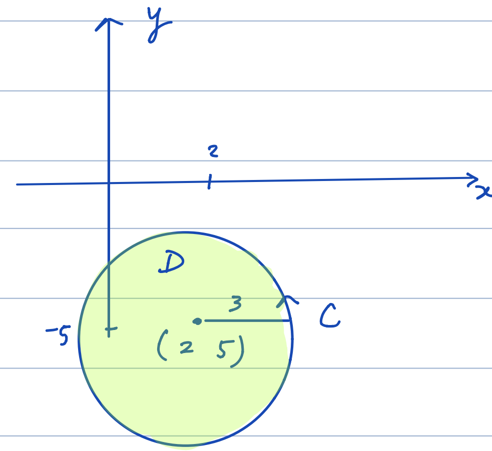
$$= -3\left(\int_0^{2\pi} d\theta\right)\left(\int_2^3 r^3\,dr\right) = -3\left(\theta\Big|_0^{2\pi}\right)\left(\frac{r^4}{4}\Big|_2^3\right)$$

$$= -3(2\pi)\left(\frac{81-16}{4}\right) = \frac{-3\pi(65)}{2} = -\frac{195\pi}{2}$$

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}$ is the vector field

$\vec{F}(x,y) = y\,\hat{i} - x\,\hat{j}$ and where C is the circle

$(x-2)^2 + (y+5)^2 = 9$



Solution :

Using Green's thm:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D (-1) - (1) \; dA = -2 \iint_D dA$$

$$= -2 \; \text{Area}(D) = -2\,(\pi (3)^2) = \boxed{-18\pi}$$

without using Green's theorem :

Parametrization of C : $\vec{r}(t) = \langle 2 + 3\cos t, -5 + 3\sin t \rangle$

$0 \le t \le 2\pi$

$\Rightarrow \; d\vec{r} = \langle -3\sin t, 3\cos t \rangle \, dt$

$$\int_C \underset{\langle y, -x \rangle}{\overset{\text{||}}{\vec{F}}} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \langle \overset{y}{-5+3\sin t}, \; \overset{-x}{-(2+3\cos t)} \rangle \cdot \langle -3\sin t, 3\cos t \rangle \, dt$$

$$= \int_0^{2\pi} 15\sin t - 9\sin^2 t - 6\cos t - 9\cos^2 t \; dt = \int_0^{2\pi} 15\sin t - 6\cos t - 9 \, dt$$

$$= -15\cos t - 6\sin t - 9t \Big|_0^{2\pi} = \boxed{-18\pi}$$