Math 2415

Problem Section #7

Make sure you do some problems from each section.

14.3-14.5: Partial Derivatives, Tangent Planes, and Chain Rule

Chain Rule: Let $z = f(x, y)$ be a function on the plane and let $(x, y) = \mathbf{r}(t)$ be a curve in the plane. The composition

$$z = g(t) = f(\mathbf{r}(t)) = f(x(t), y(t))$$

is a scalar-valued function of one variable. The function g is called the **restriction** of f to the curve $\mathbf{r}$, since it just gives us the values of f along the curve $\mathbf{r}$. In this context, the **Chain Rule for Functions on Curves** states that

$$g'(t) \;=\; \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \;=\; f_x(x(t), y(t))x'(t) \;+\; f_y(x(t), y(t))y'(t).$$

Now for the questions!

1. Let $f(x, y) = xy + x^2 = x(y + x)$ Calculate f_x, f_y, f_{xy} and f_{xx} at the point $\mathbf{x}_0 = (1, 1)$. Discuss the geometric meaning of each of these derivatives, with the aid of schematic diagrams showing slices of f in planes where x or y are constant.
2. Show that the function $u(x, y) = e^{-x} \cos(y)$ satisfies Laplace's equation $u_{xx} + u_{yy} = 0$.
3. Show that the function $u(x, t) = \cos(kx) \sin(akt)$ satisfies the wave equation $u_{tt} = a^2 u_{xx}$.
4. Find the (level set) equation of the tangent plane to the function $z = f(x, y) = e^x \cos(xy)$ at $(x_0, y_0) = (0, 0)$. Explain why your solution shows that $e^x \cos(xy) \approx x + 1$ near $(0, 0)$.
5. Let $z = f(x, y) = y^2 \sin x$ where $(x, y) = \mathbf{r}(t) = (e^{3t}, t^4)$.
   (a) Form the composition $g(t) = f(x(t), y(t))$ and then use the single variable chain rule to calculate $g'(t)$.
   (b) Use the Chain Rule for Functions on Curves to calculate $g'(t)$.
6. 14.5.7
7. 14.5.13
8. 14.5.15
9. 14.5.35

16.6, Parametrized Surfaces

1. Let S be the surface with parametrization

$$(x, y, z) = \mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\,\mathbf{k} \qquad u \geq 0, \quad 0 \leq v \leq 2\pi.$$

   (a) Show that S is a cone. **Hint:** Find an equation of the form $F(x, y, z) = 0$ for this surface by eliminating u and v from the equations for x, y, and z above.

   (b) Find a parametrization of the tangent plane to the cone at the point where $(u, v) = (2, \pi/4)$.

2. (a) Write down the equation of the form $F(x, y, z) = 0$ for the sphere of radius 2, center $(1, 2, 3)$.

   (b) Show that

$$(x, y, z) = \mathbf{r}(\theta, \phi) = (1 + 2\sin\phi\cos\theta, 2 + 2\sin\phi\sin\theta, 3 + 2\cos\phi)$$

   is a parametrization of this sphere. **Hint:** Substitute the formulae for x, y, and z in terms of θ and ϕ into the function F you obtained in (a) and simplify as much as you can. What does this calculation tell you about where each of the points $(x, y, z) = \mathbf{r}(\theta, \phi)$ lie?

3. Find a parametrization for each of the following surfaces. [Note: There are many correct answers!] Show how you arrived at your answer. **Hint:** It is often helpful to construct your parametrization using (a) cylindrical coordinates, (b) spherical coordinates, or (c) by using a parametrization such as $(x, y, z) = (u, v, f(u, v))$ for the surface $z = f(x, y)$.

   (a) The portion of the paraboloid $z = x^2 + y^2$ where $z \leq 4$.

   (b) The portion of the cone $z = 2\sqrt{x^2 + y^2}$ that is between the planes $z = 2$ and $z = 4$ and is in the first octant.