Math 2415
Problem Section #9

Make sure you do some problems from each section.
If you did not finish the problems from 14.7A last week, first do 14.7A 1a and 1b.

14.7A: Local Optimization

A guided example

Consider the function $z = f(x, y) = x^2 + 3y^2 - 2xy - 3x$. The critical points of this function are the points in the xy-plane that satisfy the equations

$$0 = \frac{\partial f}{\partial x} = 2x - 2y - 3 \quad (1)$$

$$0 = \frac{\partial f}{\partial y} = 6y - 2x. \quad (2)$$

Equations (1) and (2) are the equations of a pair of lines in the xy-plane. The critical points are the points that lie on **both** of these lines (since both equations need to hold at a critical point). By sketching the two lines you can see that in this example there can only be one critical point of f.

In general the critical point equations

$$0 = \frac{\partial f}{\partial x}(x, y) \quad (3)$$

$$0 = \frac{\partial f}{\partial y}(x, y) \quad (4)$$

are the equations of a **pair of curves** in the xy-plane. The critical points are the points that lie on both curves. Often you can sketch these two curves and use your sketch to determine how many critical points there are and roughly where they are located. Then you can use this geometric information to guide an algebraic calculation to solve Equations (3) and (4) and thereby find the precise location of the critical points. **Use this combined geometric-algebraic technique in the problems below.**

Problems

1. Find the local maxima, minima, and saddle points of the following functions
   (a) $f(x, y) = x^4 + y^2 + 2xy$
   (b) $f(x, y) = y^3 - 3y + 3x^2y$
   (c) $f(x, y) = xy + 2x + 2y - x^2 - y^2$

14.7B: Global Max/Min

1. Let D be the closed triangle with vertices $(0, 3)$, $(-3, 0)$, and $(3, 0)$. Find the absolute maximum and minimum of the function $f(x, y) = x^2 + y^2 - 2y$ on D.

2. Let D be the rectangle $[0, 4] \times [0, 2]$. Find the absolute maximum and minimum of the function $f(x, y) = 2x^2 + y^2 - 4x - 2y$ on D.

3. Let D be the half disc given by $y \geq 0$ and $x^2 + y^2 \leq 16$. Find the absolute maximum and minimum of the function $f(x, y) = x^2 + 2y^2$ on D.

14.8: The Method of Lagrange Multipliers, aka Constrained Optimization

Solution Strategy

For the problems from 14.8 you will use the method of Lagrange Multipliers to solve constrained optimization problems of the following type:

Find the absolute maximum and minimum of
$z = f(x, y)$ *subject to a constraint* $g(x, y) = k$.

In class we showed that if (x_0, y_0) is the location of such a maximum or minimum, then there is a scalar λ so that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \tag{5}$$
$$g(x_0, y_0) = k. \tag{6}$$

In other words, the triples (x_0, y_0, λ) satisfying these equations are the candidates for the solutions to our constrained optimization problem. So we call them critical points. There are two types of critical points: Those with $\lambda = 0$ and those with $\lambda \neq 0$.

For the problems below, first solve the problem geometrically and then algebraically.

For the **geometric approach**, there are two steps. In the first step, we look to see if there are any critical points with $\lambda = 0$. To do so we use the geometric approach from 14.7A to locate the points (x_0, y_0) where $\nabla f = (0, 0)$. Once you have found them, check to see if any of them lie on the constraint curve. Those that do are critical points for the constrained optimization problem. Note that most of the time there are no critical points with $\lambda = 0$, but we have to check for them just in case.

In the second step, we we look to see if there are any critical points with $\lambda \neq 0$. To do so, your task is to *visually spot those points,* (x_0, y_0)*, on the constraint curve which have the property that the tangent line to the constraint curve is equal to the tangent line to the level curve of* f *that goes through the point* (x_0, y_0). These points are also critical points for the constrained optimization problem.

By solving the problem geometrically/pictorially, you can often quickly determine the number of critical points as well as their approximate locations. In some cases it may also enable you to find the exact locations of the critical points exactly.

For the **algebraic approach** you need to solve three equations in three unknowns, (x, y, λ) for the critical points as we discussed in class. You should use the geometric solution to guide/check your algebraic calculation, and in particular to make sure you have found all the critical points.

Problems

1. Use the method of Lagrange multipliers to find the maximum and minimum of $z = f(x, y) = ye^x$ on the circle $x^2 + y^2 = 2$.

2. Use the method of Lagrange multipliers to find the maximum and minimum of $z = f(x, y) = xy^2$ on the circle $x^2 + y^2 = 1$.

   Hint: There are 6 critical points. Two of them have $\lambda = 0$ and the other four have $\lambda \neq 0$.

3. (a) Use the method of Lagrange multipliers to find the maximum and minimum of $z = f(x, y) = x^2 + y^2$ on the circle, C, given by $(x - 1)^2 + y^2 = 1$. In particular:

   (b) Explain why the solutions to this problem gives the points on the circle, C, that are closest to and furthest from the origin.

(c) Show that there are two critical points for this Lagrange multipliers problem, one of which has $\lambda = 0$.

4. In this problem you will find the absolute max and min of the function $z = f(x, y) = x^2 + y^2 - 4x - 4y$ on the closed disk $x^2 + y^2 \leq 9$.

   (a) Find critical points of f in the open disk $x^2 + y^2 < 9$.

   (b) Explain why the problem of finding the absolute maximum and minimum of the function $f(x, y) = x^2 + y^2 - 4x - 4y$ on the circle $x^2 + y^2 = 9$ is the same as the problem of finding the absolute maximum and minimum of the function $h(x, y) = 9 - 4x - 4y$ on the circle $x^2 + y^2 = 9$.

   (c) To solve the problem of optimizing the function $z = h(x, y)$ on the circle $x^2 + y^2 = 9$ you can just use the geometric approach outlined above. The symmetry of the problem will enable you to find the exact locations of the critical points!