1 Chapter 4. Calculus on Surfaces

For \( \vec{x} : D \in \mathbb{R}^2 \to \mathbb{R}^3 \) to parameterize a surface \( M \), so we can do calculus properly on \( M \) we need that:

- The grid curves form a coordinate system on \( M \)
- There is a tangent space to \( M \) at each point of \( M \). Note \( \vec{x}^* (U_1) = \partial \vec{x}/\partial u = \vec{x}_u \)
- Later we will see that tp gaurantee these two requirements hold it is enough that \( \{ \vec{x}^* (U_1), \vec{x}^* (U_2) \} \) is a linearly independent set. In this case, the tangent space will be defined to be the 2D vector space \( T_{\vec{x}(p)}M = \text{Span}\{ \vec{x}^* (U_1), \vec{x}^* (U_2) \} \subset T_{\vec{x}(p)}\mathbb{R}^3 \).

**Definition 1.1.** \( \vec{x} : D \in \mathbb{R}^2 \to \mathbb{R}^3 \) is a regular mapping if \( \forall \vec{p} \in D \), the tangent mapping \( \vec{x}^* : T_p\mathbb{R}^2 \to T_{\vec{x}(p)}\mathbb{R}^3 \) satisfies one of the following equivalent conditions:

- \( \{ \vec{x}^* (U_1), \vec{x}^* (U_2) \} \) is a linearly independent set
- \( \vec{x}_u \times \vec{x}_v \neq \vec{0} \)
- \( \vec{x}^* \) is one-to-one
- Let \( D\vec{x} = [\partial \vec{x}/\partial u, \partial \vec{x}/\partial v] \) be the 3 x 2 matrix of partial derivatives of \( \vec{x} \). Then, \( \text{rank}(D\vec{x}) = 2 \).

**Definition 1.2.** (Coordinate Patch)

- A coordinate patch is a mapping \( \vec{x} : D \in \mathbb{R}^2 \to \mathbb{R}^3 \) that is one-to-one and regular on an open set \( D \in \mathbb{R}^2 \)
- A set \( D \in \mathbb{R}^2 \) is open if \( \forall \vec{x}_0 \in D \), there exists \( \epsilon > 0 \) so that open ball radius \( \epsilon \), centered at \( \vec{x}_0 \) lies entirely in \( D \). (this ball is the set \( \{ \vec{x} \in \mathbb{R}^2 \mid ||\vec{x} - \vec{x}_0|| < \epsilon \} \).
- The image \( M = \vec{x}(D) \) of a coordinate patch is an example of a surface.

**Notes:**

- We need \( \vec{x} \) regular to ensure tangent space to \( M \) exists and is 2D
• We need \( \vec{x} \) one-to-one to rule out self intersections in \( M \)

• Then \( M \) looks locally like a plane and we can use \( \vec{x} \) to turn calculus problems on \( M \), into calculus problems on \( D \in \mathbb{R}^2 \), which are easy.

Example Graph of a function \( z = f(x, y) \). For example,

\[
z = f(x, y) = 3x^2 + 4y^2
\]

which is a elliptic paraboloid. A Monge Patch is a parametrization of \( z = f(x, y) \) of the form \( \vec{x}(u, v) = (u, v, f(u, v)) = (x, y, z) \) (graph in hand written notes). We have that

\[
D\vec{x} = \begin{pmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v}
\end{pmatrix}
\]

the columns of \( D\vec{x} \) are linearly independent regardless of \( f \) since there are 2 pivots.

Notice. For any coordinate patch \( \vec{x} : D \in \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) there is an inverse mapping \( \vec{x}^{-1} : M = \vec{x}(D) \rightarrow D \) given by \( \vec{x}^{-1}(x, y, z) = (u, v) \), where \( \vec{x}(u, v) = (x, y, z) \). The inverse map, \( \vec{x}^{-1} \), is well defined as \( \vec{x} \) is one-to-one.

For a Monge Patch \( \vec{x}^{-1} \) is continuous. But there exists coordinate patches for which \( \vec{x}^{-1} \) is not continuous. In order to avoid self intersections in \( M \) and to ensure the calculus calculations we do on \( (u, v) \) space can be correctly interpreted on \( M \) we also need the patch to be proper.

Definition 1.3. A coordinate patch is called \textbf{proper} if \( \vec{x}^{-1} \) is continuous.

Definition 1.4. A surface in \( \mathbb{R}^3 \) is a subset \( M \in \mathbb{R}^3 \) so that for every point \( p \in M \) there is a proper patch \( \vec{x} : D \rightarrow \mathbb{R}^3 \) whose image \( \vec{x}(D) \) is a subset of \( M \) that contains a neighborhood \( N \) of \( p \).

More Examples

Geographical patch on the sphere, \( S^2 \).

\[
\vec{x}(u, v) = (r \cos v \cos u, r \cos v \sin u, r \sin v)
\]

where \( -\pi < u < \pi \) and \( -\pi/2 < v < \pi/2 \). \( u \) is angle in \( xy \)-plane from \( x \)-axis and \( v \) is angle above equator. This patch covers all of sphere except north and south poles and a circle of longitude on the back side of the sphere in the \( xz \)-plane (where \( x < 0 \)). Taking a cross product you can check the patch is regular. You can also check it is 1-1 and proper.