Reviews

The question whether mathematics is discovered or created divides the mathematical community into two camps. Some mathematicians—like me—are in one camp in the morning and in the other one in the afternoon. My opinion, I confess, depends on the type of work being done at the moment. But deep in my mind, I am fully convinced that, based on some very elementary and not yet understood endowment of our brain, the fantastic mathematical universe is human-made. This can't be proved mathematically. The best one can hope for are compelling arguments and strong empirical evidence.
This is what Klaus Truemper's book "The Construction of Mathematics: The Human Mind's Greatest Achievement" delivers. It sheds surprising and fascinating new light on the issue. Powerful arguments are provided by using the method of language games invented by the philosopher Ludwig Wittgenstein. Employing results of modern brain science about human cognition, the book also explains how it is possible that eminent mathematicians and scientists arrive at diametrically opposed answers for the creation vs. discovery question. Truemper's findings are consistent with my ultimate conviction.

—Martin Grötschel, mathematician and President, Berlin-Brandenburg Academy of Sciences and Humanities, Germany

Klaus Truemper has made an original and daring attack on the foundations of mathematics. Readers will enjoy his forthright and unswerving analysis. His ideas should become recognized and influential.

—Reuben Hersh, mathematician and award-winning author of a number of books on the nature, practice, and social impact of mathematics

As computational methods become increasingly powerful, the engineer of today often resorts to simulation methods and is acutely aware of the limitations of contrived analytical mathematical methods. Truemper's exposition puts into focus the debate as to whether mathematics is really intrinsic to the physical world or is in fact made up as we go along. The elucidation the book delivers on this topic is of significance, as science often moves along faster when we are released from anachronistic notions that imprison freedom of thought. Klaus Truemper permits us to be more daring with our mathematics.

—Derek Abbott, physicist and engineer, University of Adelaide, Australia

Is mathematics the product of human creativity and ingenuity or does it exist independently of mankind and only waits to be discovered? Is the latter the reason for its incredible success or is mathematics not so indispensable in our world after all?

After a fascinating tour through the history of mathematics and computation that provides astonishing new perspectives, Klaus Truemper's book addresses the philosophical question of creation versus discovery from many different directions ranging from Wittgenstein's philosophy to brain science. Pros and cons are collected and discussed in a comprehensive and—given the difficult subject—remarkably light and entertaining way and: the book comes to a conclusion.

Is the issue finally settled? No! Can it possibly be? Most certainly not! The given circumstantial evidence does, however, stimulate the reader to rethink his or her point of view, question his or her own arguments, rethink the given reasoning and sharpen the issue. The book is a pleasure to read!

—Peter Gritzmann, mathematician, author of mathematics and popular science books

This wonderful book addresses the oldest and thorniest question in the philosophy of mathematics: Is mathematics discovery or invention? The key contribution of this book is to use Ludwig Wittgenstein's technique of language games to shed light on this deep philosophical question. Overall this is the most insightful and com-

pelling contribution to the debate that I have read and I am inclined to agree with the conclusions.
But the book is much more than an important contribution to a fundamental debate. It is beautifully and lucidly written and, in contrast to most texts on philosophical matters, is enjoyable and easy to read. Mathematicians know that a picture is worth a thousand words, and the rich use of beautifully presented illustrations genuinely enriches the readers experience.
The book begins with a short history of the major themes of mathematics. It is, by a country mile, the most insightful such history that I have read. An intelligent reader with no interest whatsoever in philosophy would still learn much from this book. I cannot recommend it too highly.

—Geoffrey Whittle, mathematician, Victoria University of Wellington, New Zealand

Klaus Truemper's "The Construction of Mathematics: The Human Mind's Greatest Achievement" is a unique blend of history and penetrating philosophical analysis. After taking the reader on a dizzying journey through the history of mathematics and computation, Truemper arrives at his central question: Is mathematics discovered or invented?
Relying on a range of philosophical approaches and some brilliant argumentation, he arrives at what seems like the only possible answer. Mathematics, he shows, is not "out there," waiting to be discovered; it is, rather, the highest creation of the human mind.
Truemper's book is not only insightful and original but also fast-paced and gripping. Whether you are a mathematician, historian, philosopher, or layman, you will find it thought-provoking as well as highly enjoyable.

—Amir Alexander, historian of science and award-winning author of books on the interconnection of mathematics and its social, cultural, and political settings

THE CONSTRUCTION OF MATHEMATICS

THE HUMAN MIND'S GREATEST ACHIEVEMENT

KLAUS TRUEMPER

Leibniz Company

Hardcover published by Leibniz Company
2304 Cliffside Drive
Plano, Texas, 75023
USA

Original edition 2017
Updated edition 2018
Second updated edition 2019

Cover Art:
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The book is typeset in LaTeX using the Tufte-style book class, which was inspired by the work of Edward R. Tufte and Richard Feynman.

Sources and licensing arrangements for all figures are included in the Notes section.

Library of Congress Cataloging-in-Publication Data
Truemper, Klaus, 1942–

The Construction of Mathematics: The Human Mind's Greatest Achievement
Includes bibliographical references and subject index.
ISBN 978-0-9663554-8-2
1. Mathematics. 2. Construction

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Introduction

Over the span of tens of thousands of years, humans have created an elaborate body of theory unequaled in size and complexity: mathematics. Early work on that edifice was slow, with the pace of progress measured in thousands of years. But gradually things speeded up; progress was achieved in hundreds of years, then in decades. And now the theory is leaping forward at a dizzying pace.

There are hundreds of books chronicling that development either in multi-volume works attempting to cover the entire spectrum of results, or in more specialized texts investigating specific areas or periods.[1] Amazingly, some authors—for example Florian Cajori (1859–1930) and James Roy Newman (1907–1966)—accomplished both goals.[2]

This book is different. It poses questions, evaluates answers, and develops conclusions about mathematics as a creation of the human mind.

Chapters 2–7 provide a longitudinal overview of important parts of mathematics that, by their very nature, can be covered in everyday language and appealing simplicity.

The key idea is to bring out the struggle for insight in the face of very difficult questions; the paths of investigation that often proved to be correct, but sometimes to be dead ends; and the triumphant clarity eventually achieved.

There is a profound philosophical question: Where do all the results of this gigantic body of theory come from? Are they already present in some hidden, possibly metaphysical, location and then discovered by inquisitive minds, or are they created in the same way that engineers design various machines for energy conversion, production of goods, or transportation?

Chapter 8 reviews this question and prior answers. They claim either creation, or discovery, or some combination of both. Obviously, many of these answers are in conflict. Yet each of them is defended with seemingly irrefutable arguments.[3] In the second part of the book, we investigate this question and work toward a resolution.

Chapters 9 and 10 introduce and then use the concept of language games first proposed by the philosopher Ludwig Wittgenstein (1889–1951) for the resolution of philosophical problems. A *language game* is a controlled setting of language use that brings a particular facet of a given philosophical problem into focus. One then imagines the operation of the language game and thus gains insight into that facet.

For the identification of relevant facets and the subsequent construction of the corresponding language games, one initially assembles a large number of example situations involving the given philosophical problem. For the case at hand, Chapters 2–6 supply many example situations. From these, we identify a number of facets, then build and operate a language game for each of them.

In some of the language games, mathematical developments are linked with creative processes of the arts, in particular sculpture and music composition.

Other language games involve imagined events, processes, or dialogues. During the operation of each language game, we always assume that mathematics is discovered. It turns out that the operation of each language game then produces some contradictory conclusion, indicating that the assumption of discovery is flawed.

Chapters 11 and 12 look at particular arguments in favor of the discovery claim. A strong argument for discovery is based on the seemingly astonishing agreement of mathematics and nature. For example, Einstein's mathematical theory of relativity explains how space and time of nature are linked. Such universal agreement is only possible if mathematics is part of nature—so the argument goes—and thus mathematics can only be discovered.

We examine that argument and see that a different interpretation is possible: We have *shaped* mathematics so that we can represent processes of nature. Moreover, we have *defined* the processes supposedly occurring in nature. It's no wonder that *our* mathematics can represent *our* processes!

There also are many more stories of failed mathematical models versus successful ones, and we have a tendency to ignore, gloss over, or simply forget those failures.

Then there are processes of nature—of course, as we have defined them—where we have not been able to build any mathematical model or carry out certain computations. For those cases, mathematical theories have been built that rule out such model building or computation. How could these theories be part of nature?

Yet another aspect is the following. There are mathematical theories that are in blatant violation of the processes of nature as we have defined them. How can these theories be part of nature?

Another argument in favor of discovery is that humans simply cannot live without mathematics, and thus that mathematics must be part of nature. For a counterargument to this claim, we look at a community of humans that live successfully and happily without any mathematics at all.

For more than 30 years, we have repeatedly posed the question of creation versus discovery of mathematics to mathematicians. Amazingly, the vast majority voted for discovery. We say "amazingly," since opinions of eminent mathematicians of the 19th and early 20th century were mixed. For example, Georg Cantor (1845–

1918), Richard Dedekind (1831–1916), and Carl Friedrich Gauss (1777–1855) argued for creation, while Gottlob Frege (1848–1925) and Kurt Gödel (1906–1978) claimed that discovery takes place. How can these differences be explained, in particular the shift toward discovery?

For an answer, Chapter 13 takes a critical look at the brain, which after all is the machinery that makes and evaluates the arguments swirling around the question.

Brain science has made incredible strides since the 1990s. In this book we use the new results and investigate how the structure of the brain may impact our answers. In the process, we formulate a conjecture that sheds light on the reason for the different opinions.

Chapter 14 summarizes the various arguments. In the face of so many contradictions arising from the discovery hypothesis, we come to the conclusion that mathematics is indeed created by the human mind. That conclusion fits the historical facts about mathematical developments, does not require metaphysical concepts, and is consistent with the view that the human activities of music composition, sculpture, and writing are creation. The conclusion is also consistent with the notion of Occam's razor[4] of science, according to which the simpler explanation is preferred when there is a choice.

Above all, reading the book is meant to be an enjoyable journey across a wide-ranging territory of human thought and accomplishment. If you agree in the end that this goal has been achieved, we have done our job.

Technical Remarks

Chapters 2–14 don't require any mathematical background beyond everyday knowledge of numbers and the elementary operations of addition, subtraction, multiplication, and division. The subsequent, extensive, Notes section expands upon the discussion and justifies various statements using at most high school mathematics. The Notes section also points to references and material readily available on the Internet, and it lists the licenses under which the various figures are included in the book. The reader may choose to ignore the latter notes since they are only of legal interest.

Portraits are included for virtually all mathematicians mentioned in the book, courtesy of Wikipedia and various academic and research institutions. The portrait appears on first mention of each mathematician.