The denotational semantics of loops is part of a more general mathematical theory of complete partial orders and continuous functions. Some of the basics of that theory are presented below, culminating in the Knaster-Tarski Fixed-Point Theorem. We use the Fixed-Point Theorem to prove that our denotational definition of \texttt{while} loops is a well-formed mathematical definition and constitutes the least fixed point of the functional $\Gamma$. We begin with important definitions.

**Definition:** A partial order (p.o.) is a set $P$ on which there is a binary relation $\sqsubseteq$ which is

(i) reflexive: $\forall p \in P . p \sqsubseteq p$,

(ii) transitive: $\forall p, q, r \in P . (p \sqsubseteq q) \land (q \sqsubseteq r) \Rightarrow (p \sqsubseteq r)$, and

(iii) antisymmetric: $\forall p, q \in P . (p \sqsubseteq q) \land (q \sqsubseteq p) \Rightarrow (p = q)$.

**Definition:** A p.o. $(P, \sqsubseteq)$ has a bottom element $\bot_P$ iff there exists an element $\bot_P \in P$ such that for all $p \in P$, $\bot_P \sqsubseteq p$.

Observe that $(\Sigma \Rightarrow \Sigma, \subseteq)$ is a partial order because the subset relation $\subseteq$ is reflexive, transitive, and antisymmetric. The empty set $\emptyset$ (i.e., the partial function that is undefined for all inputs) is a bottom element of this partial order because the empty set is a subset of every set.

**Definition:** We say $p \in P$ is an upper bound of a subset $X \subseteq P$ iff $\forall q \in X . q \sqsubseteq p$.

Note that not every set of partial functions has an upper bound. For example, if $f(\sigma) \neq g(\sigma)$, then the set $\{f, g\}$ has no upper bound because there is no function $h$ such that $f \subseteq h$ and $g \subseteq h$. However, for any two partial functions such that $f \subseteq g$, $g$ is an upper bound of $\{f, g\}$.

**Definition:** We say $p$ is a least upper bound of $X$, written $p = \bigsqcup X$, if $p$ is an upper bound of $X$ and $p \subseteq q$ for all upper bounds $q$ of $X$. We also denote the least upper bound of two elements $p, q \in P$ as $p \sqcup q$.

In the above example, $g$ is also a least upper bound for $\{f, g\}$ because $g = f \sqcup g$.

**Definition:** An $\omega$-chain of a partial order $(P, \sqsubseteq)$ is an infinite sequence $p_0, p_1, \ldots \in P$ such that $p_0 \sqsubseteq p_1 \sqsubseteq \cdots$.

Recall that we proved in class that $\bot \subseteq \Gamma(\bot) \subseteq \Gamma^2(\bot) \subseteq \cdots$ is a family of nested subsets. Therefore, $\bot, \Gamma(\bot), \Gamma^2(\bot), \ldots$ is an $\omega$-chain for $(\Sigma \Rightarrow \Sigma, \subseteq)$. 
Theorem. Functional $\Gamma$ is continuous.

The proof is simple, and is left as an exercise to the reader.

**Definition:** A partial order $(P, \subseteq)$ is a complete partial order (cpo) iff every $\omega$-chain $p_0, p_1, \ldots \in P$ has a least upper bound $\bigcup_{i \geq 0} p_i \in P$.

Observe that $(\Sigma \rightarrow \Sigma, \subseteq)$ is a cpo because for every $\omega$-chain, the infinite union of all partial functions in the chain is also a partial function in $\Sigma \rightarrow \Sigma$. That infinite union is a least upper bound of the chain. For example, $\bigcup_{i \geq 0} \Gamma^i(\bot)$ is a least upper bound for the chain $\bot, \Gamma(\bot), \Gamma^2(\bot), \ldots \in \Sigma \rightarrow \Sigma$.

**Definition:** A function $f : P \rightarrow P$ is monotonic iff for all $p, q \in P$, $p \sqsubseteq q \implies f(p) \sqsubseteq f(q)$.

**Theorem.** Functional $\Gamma$ is monotonic.

The proof is simple, and is left as an exercise to the reader.

**Definition:** A function $f : P \rightarrow P$ is continuous iff it is monotonic and for all $\omega$-chains $p_0, p_1, \ldots \in P$, we have $\bigcup_{i \geq 0} f(p_i) = f\left( \bigcup_{i \geq 0} p_i \right)$.

**Theorem.** Functional $\Gamma$ is continuous.

**Proof.** Let $p_0, p_1, p_2, \ldots \in \Sigma \rightarrow \Sigma$ be an arbitrary $\omega$-chain in cpo $(\Sigma \rightarrow \Sigma, \subseteq)$. The proof that $\Gamma$ is continuous consists of two parts: First we prove that if $(\sigma, \sigma') \in \bigcup_{i \geq 0} \Gamma(p_i)$ then $(\sigma, \sigma') \in \Gamma\left( \bigcup_{i \geq 0} p_i \right)$. This proves that $\bigcup_{i \geq 0} \Gamma(p_i) \subseteq \Gamma\left( \bigcup_{i \geq 0} p_i \right)$. Next we prove that if $(\sigma, \sigma') \in \Gamma\left( \bigcup_{i \geq 0} p_i \right)$ then $(\sigma, \sigma') \in \bigcup_{i \geq 0} \Gamma(p_i)$. This proves that $\bigcup_{i \geq 0} \Gamma(p_i) \supseteq \Gamma\left( \bigcup_{i \geq 0} p_i \right)$. We conclude therefore that $\bigcup_{i \geq 0} \Gamma(p_i) = \Gamma\left( \bigcup_{i \geq 0} p_i \right)$.

**Proof of $\subseteq$ direction:** Let $(\sigma, \sigma') \in \bigcup_{i \geq 0} \Gamma(p_i)$ be given. Thus, there exists $n \geq 0$ such that $(\sigma, \sigma') \in \Gamma(p_n)$. Since $p_n \subseteq \bigcup_{i \geq 0} p_i$, it follows from the monotonicity of $\Gamma$ that $\Gamma(p_n) \subseteq \Gamma\left( \bigcup_{i \geq 0} p_i \right)$. Therefore $(\sigma, \sigma') \in \Gamma\left( \bigcup_{i \geq 0} p_i \right)$.

**Proof of $\supseteq$ direction:** Now instead let $(\sigma, \sigma') \in \Gamma\left( \bigcup_{i \geq 0} p_i \right)$ be given. From the definition of $\Gamma$, we know there are two possible cases:

- **Case 1:** If $B[b]\sigma = F$ then $\sigma' = \sigma$. Since $\{ (\sigma, \sigma') \mid B[b]\sigma = F \}$ is a subset of $\Gamma(x)$ for every set $x$, it follows that $(\sigma, \sigma') \in \bigcup_{i \geq 0} \Gamma(p_i)$.

- **Case 2:** If $B[b]\sigma = T$ then $\sigma' = \left( \bigcup_{i \geq 0} p_i \right) \Gamma(c)\sigma$. Thus, $(\Gamma(c)\sigma, \sigma') \in p_n$, so there exists $n \geq 0$ such that $(\Gamma(c)\sigma, \sigma') \in p_n$. Since $B[b]\sigma = T$, it follows from the definition of $\Gamma$ that $(\sigma, \sigma') \in \Gamma(p_n)$. We conclude that $(\sigma, \sigma') \in \bigcup_{i \geq 0} \Gamma(p_i)$.

**Definition:** Let $f : P \rightarrow P$ be a continuous function on a cpo $P$. A fixed point of $f$ is an element $p \in P$ such that $f(p) = p$.

**Theorem** (Knaster-Tarski Fixed-Point Theorem): Let $f : P \rightarrow P$ be a continuous function on a cpo $P$ with bottom $\bot$. Then $\bigcup_{i \geq 0} f^i(\bot)$ is a least fixed point of $f$.

From the fixed-point theorem we conclude that $\bigcup_{i \geq 0} \Gamma^i(\bot)$ is a least fixed point of $\Gamma$. 

2