# CS 4349.003 Homework 10

Due Wednesday November 20th on eLearning

November 13, 2019

Please answer the following **2** questions.

To give everybody a bit a breather, **there will be no late penalty for submissions made before 1:00pm on Friday, November 22nd**. (It's up to you if you want to keep doing homework right up to the beginning of Fall Break.) As usual, submissions after 1:00pm on Friday may not be accepted so we can release solutions.

1. Consider the directed graph $G = (V, E)$ below with non-negative capacities $c : E \to \mathbb{R}_{\geq 0}$ and an $(s,t)$-flow $f : E \to \mathbb{R}_{\geq 0}$ that is feasible with respect to $c$. Each edge is labeled with its flow/capacity.

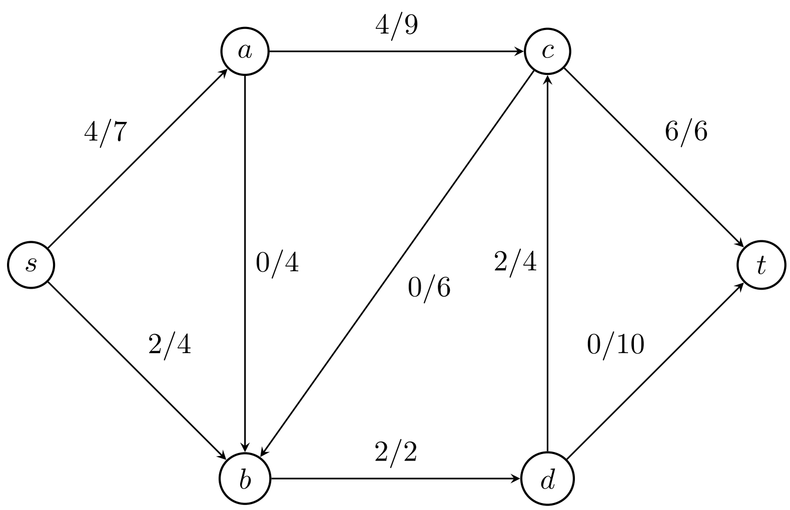


**Figure 1.** An $(s,t)$-flow $f$. The number to the left of each '/' is the current flow on the edge. The number of the right of the '/' is the capacity of the edge.

   (a) Draw the residual graph $G_f = (V, E_f)$ for flow $f$. Be sure to label every edge of $G_f$ with its residual capacity.

   (b) Describe an augmenting path $s = v_0 \to v_1 \to \ldots \to v_r = t$ in $G_f$ by either drawing the path in your residual graph or listing the path's vertices in order.

   (c) Let $F = \min_i c_f(v_i \to v_{i+1})$ and let $f' : E \to \mathbb{R}_{\geq 0}$ be the flow obtained from $f$ by pushing $F$ units through your augmenting path. Draw a new copy of $G$, and label its edges with the flow values for $f'$.

(d) Draw the residual graph $G_{f'} = (V, E_{f'})$ for flow $f'$.

(e) There shouldn't be any augmenting paths in $G_{f'}$, implying $f'$ is a maximum flow. Draw or list the vertices in $S$ for some minimum $(s,t)$-cut $(S,T)$.

(f) What is the value of the maximum flow/capacity of the minimum cut?

2. Consider the following generalization of the bipartite matching problem. You are given an undirected bipartite graph $G = (L \cup R, E)$ where $L \cap R = \emptyset$ and every edge connects a vertex in $L$ to a vertex in $R$. You are also given a set of non-negative integer ***limits*** $\ell : (L \cup R) \to \mathbb{Z}_{\geq 0}$. Describe and analyze an algorithm that returns a maximum size subset of edges $E' \subseteq E$ such that each vertex $v$ is incident to *at most* $\ell(v)$ edges in $E'$. As usual, you may express your running time in terms of $V$ and $E$, the number of vertices and edges in $G$.

3. **(Extra credit)** Suppose you are given a flow network $G$ with *integer* edge capacities and an *integer* maximum flow $f^*$ in $G$. Describe algorithms for the following operations:

   (a) INCREMENT($e$): Increase the capacity of edge $e$ by 1 and update the maximum flow.

   (b) DECREMENT($e$): Decrease the capacity of edge $e$ by 1 and update the maximum flow.

   Both algorithms should modify $f^*$ so that it is still a maximum flow under the new capacities more quickly than recomputing a maximum flow from scratch.