CS 4349.400 Homework 1

Due Wednesday September 5th, in class

August 27, 2017

Please answer each of the following questions.

Some important homework policies

- Each group of up to three students must write their solutions in their own words and submit their solutions on paper at the beginning of class. Clearly print the names of every member of your group, the homework number (Homework 1), and the problem number at the top of every page. Start each numbered homework problem on a new sheet of paper, and staple your entire assignment together.
- Unless the problem states otherwise, you must justify (prove) that your solution is correct.
- We strongly suggest you use LaTeX to typeset your solutions. Any illegible solutions will be considered incorrect. The announcement for this homework links to a template for writing solutions in LaTeX.
- If you use outside sources or write solutions in close collaboration with anybody outside your group, then you may cite that source or person and still receive full credit for the solution. Material from the lecture, the required textbook, or prerequisite courses need not be cited. Failure to cite other sources or failure to provide solutions in your own words, even if quoting a source, is considered an act of academic dishonesty.
- Writing "I don't know" ***and nothing else*** for any question or lettered part of a question is worth 25% credit. If you leave the solution blank or write anything else, we will grade exactly what is written.
- Some problems on this and future homework may require skills from previous courses or education levels such as the ability to do algebraic manipulation on logarithms or exponents, the ability to evaluate arithmetic, geometric, or other simple series, or the ability to use basic data structures and recall their asymptotic performances.

See https://utdallas.edu/~kyle.fox/courses/cs4349.004.18f/about.shtml and https://utdallas.edu/~kyle.fox/courses/cs4349.004.18f/writing.shtml for more detailed policies. If you have any questions about these policies, please do not hesitate to ask in class, in office hours, or through email.

1. The nth *Fibonacci binary tree* $\mathscr{F}_n$ is defined recursively as follows.

   - $\mathscr{F}_1$ is a single root node with no children.
   - For all $n \geq 2$, $\mathscr{F}_n$ is obtained from $\mathscr{F}_{n-1}$ by adding a right child to every leaf and adding a left child to every node that has only one child.

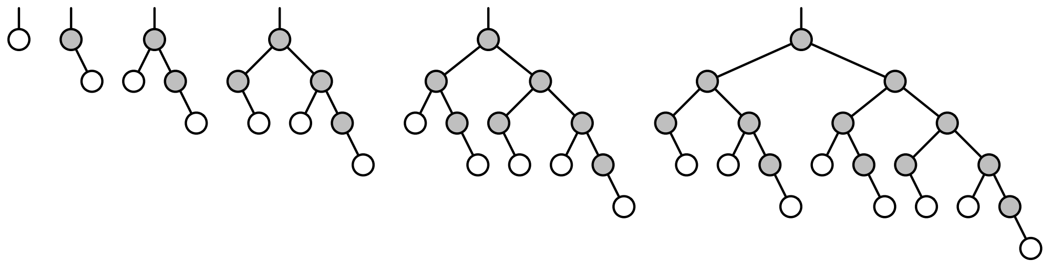

Figure 1. The first six Fibonacci binary trees. In each tree $\mathscr{F}_n$, the subtree of gray nodes is $\mathscr{F}_{n-1}$.

   (a) Truthfully write the phrase ***"I have read and understand the course policies."***

   (b) Prove that the number of leaves in $\mathscr{F}_n$ is precisely the nth Fibonacci number: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$.

   (c) How many nodes does $\mathscr{F}_n$ have? Given an exact closed-form answer in terms of Fibonacci numbers, and prove your answer is correct.

   (d) Prove that for all $n \geq 2$, the right subtree of $\mathscr{F}_n$ is a copy of $\mathscr{F}_{n-1}$.

2. Sort the functions listed below from asymptotically smallest to asymptotically largest, indicating ties if there are any. **Do not turn in proofs for this problem.** (But you may want to write the proofs for yourselves anyway so you know if you're correct.) To simplify your answers, write $f(n) \ll g(n)$ to mean $f(n) = o(g(n))$, and write $f(n) \equiv g(n)$ to mean $f(n) = \Theta(g(n))$. For example, the functions $n^2, n, \binom{n}{2}, n^3$ could be sorted either as $n \ll n^2 \equiv \binom{n}{2} \ll n^3$ or $n \ll \binom{n}{2} \equiv n^2 \ll n^3$.

2^n	n^2	$\lg\lg n$	n	$\lg n$	$\ln n$	$\sqrt{n}$	$n \log n$
9^n	$n^{2.5}$	$\lg^2 n$	$9n^2$	$n^2 - 500n$	$2 + \cos n$	H_n	$H_{\sqrt{n}}$
$2^{3\lg n}$	$(1 + \frac{1}{n})^n$	$\lg 3n$	$\lg^{0.3} n$	e^n	500	$\lg^{0.5} n$	$9^{\lg n}$

   As a reminder, we use the following conventions:

   - $\lg n = \log_2 n \neq \ln n = \log_e n$
   - $\lg^3 n = (\lg n)^3 \neq \lg\lg\lg n$
   - The nth harmonic number is $H_n = \sum_{i=1}^{n} 1/i \approx \ln n + 0.577215\ldots$

3. "The Barley Mow" is a cumulative drinking song which has been sung throughout the British Isles for centuries. The song has many variants, but one version traditionally sung in Devon and Cornwall has the following pseudolyrics, where $vessel[i]$ is the name of a vessel. The traditional song uses the following vessels: nipperkin, gill pot, half-pint, pint, quart, pottle, gallon, half-anker, anker, firkin, half-barrel, barrel, hogshead, pipe, well, river, and ocean. (Every vessel in this list is twice as big as its predecessor, except that a firkin is actually 2.25 ankers, and the last three units are just silly.)

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BarleyMow(vessels[1..n]):
  "Here's a health to the barley-mow, my brave boys,"
  "Here's a health to the barley-mow!"

  "We'll drink it out of the jolly brown bowl,"
  "Here's a health to the barley-mow!"
  "Here's a health to the barley-mow, my brave boys,"
  "Here's a health to the barley-mow!"

  for i ← 1 to n
      "We'll drink it out of the vessel[i], boys,"
      "Here's a health to the barley-mow!"
      for j ← i down to 1
          "The vessel[j],"
      "And the jolly brown bowl!"
      "Here's a health to the barley-mow!"
      "Here's a health to the barley-mow, my brave boys,"
      "Here's a health to the barley-mow!"
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(a) Suppose each name $vessel[i]$ is a single word, and you can sing four words a second. How long would it take you to sing BarleyMow($vessels[1..n]$)? (Give a tight asymptotic bound in n using Θ-notation.)

(b) If you want to sing this song for arbitrarily large values of n, you'll have to make up your own vessel names. To avoid repetition, these names must become progressively longer as n increases. ("We'll drink it out of the hemisemidemiyottapint, boys!") Suppose $vessel[i]$ has $\Theta(\log i)$ syllables, and you can sing six syllables per second. Now how long would it take you to sing BarleyMow($vessels[1..n]$)? (Give a tight asymptotic bound in n using Θ-notation.)

(c) Suppose each time you mention the name of a vessel, you actually drink the corresponding amount of beer:[1] one ounce for the jolly brown bowl, and 2^i ounces for each $vessel[i]$. *Exactly* how many ounces of beer would you drink if you sang BarleyMow($vessels[1..n]$)? (Try to give an *exact* answer, not just an asymptotic bound. A tight asymptotic bound, with justification, is worth significant partial credit.)

[1]For the purposes of this problem, you may substitute in your preferred beverage or assume you are at least 21 years of age.