CS 4349.400 Homework 8

Due Wednesday November 14th, in class

Please answer each of the following questions.

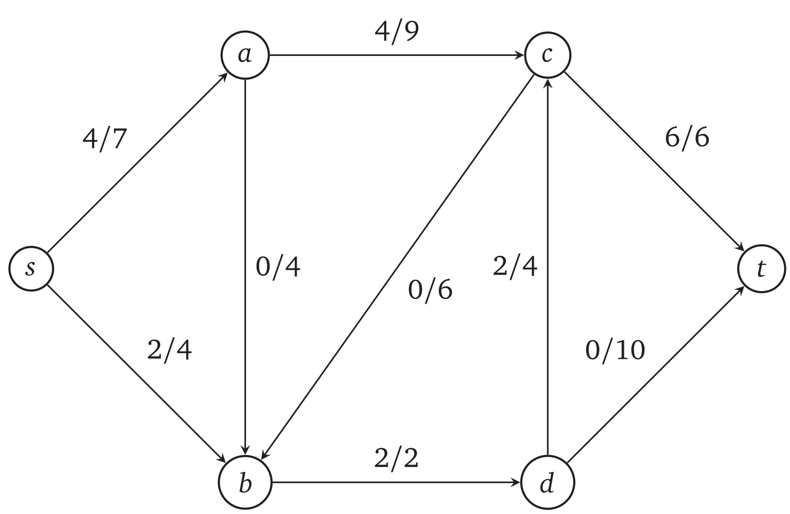
1. Let $G = (V, E, w)$ be a directed graph with weighted edges $w : E \to \mathbb{R}$; edge weights could be positive, negative, or zero. We're going to design another algorithm for computing all-pairs shortest paths. For simplicity, you may assume G is complete, meaning $E = V \times V$.

   (a) Let v be an arbitrary vertex of G. Describe an algorithm that constructs a directed graph $G' = (V', E', w')$ with edges weights $w' : E' \to \mathbb{R}$, where $V' = V \setminus \{v\}$, and the shortest-path distance between any two nodes in G' is equal to the shortest-path distance between the same two nodes in G. The algorithm should run in $O(V^2)$ time. *[Hint: When should $w'(u{\to}w) \neq w(u{\to}w)$?]*

   (b) Now suppose we have already computed all shortest-path distances in G'. Describe an algorithm to compute the shortest-path distances in the original graph G from v to every other vertex, and from every vertex to v, all in $O(V^2)$ time. *[Hint: Guess the first/last edge on each path.]*

   (c) Combine parts (a) and (b) into another all-pairs shortest path algorithm. Your algorithm should run in $O(V^3)$ time. *[Hint: Recursion.]*

2. Consider the directed graph $G = (V, E)$ below with non-negative capacities $c : E \to \mathbb{R}_{\geq 0}$ and an (s, t)-flow $f : E \to \mathbb{R}_{\geq 0}$ that is feasible with respect to c. Each edge is labeled with its flow/capacity.



An (s, t)-flow f. Each edge is labeled with its flow/capacity.

(a) Draw the residual graph $G_f = (V, E_f)$ for flow f. Be sure to label every edge of G_f with its residual capacity.

(b) Describe an augmenting path $s = v_0 \to v_1 \to \ldots \to v_r = t$ in G_f by either drawing the path in your residual graph or listing the path's vertices in order.

(c) Let $F = \min_i c_f(v_i \to v_{i+1})$ and let $f' : E \to \mathbb{R}_{\geq 0}$ be the flow obtained from f by pushing F units through your augmenting path. Draw a new copy of G, and label its edges with the flow values for f'.

(d) Draw the residual graph $G_{f'} = (V, E_{f'})$ for flow f'.

(e) There shouldn't be any augmenting paths in $G_{f'}$, implying f' is a maximum flow. Draw or list the vertices in S for some minimum (s, t)-cut (S, T).

(f) What is the value of the maximum flow/capacity of the minimum cut?

3. Suppose we are given a flow network $G = (V, E)$ in which every edge has capacity 1, together with an integer $k \geq 0$. Describe an algorithm to identify k edges in G such that after deleting those k edges, the value of the maximum (s, t)-flow in the remaining graph is as small as possible. *[Hint: The value of the maximum (s, t)-flow is equal to the capacity of the minimum (s, t)-cut.]*