Single source shortest paths

Given directed graph $G=(V,E)$

& edge weights $w: E \to \mathbb{R}^+$

a source vertex $s \in V$.

Goal: Compute shortest paths from s to other vertices.

For each $v \in V$, we have

- $dist(v)$: pessimistic guess on distance from s to v
- $pred(v)$: predecessor of v in a tentative shortest walk from s to v

```
INITSSSP(s):
  dist(s) ← 0
  pred(s) ← NULL
  for all vertices v ≠ s
      dist(v) ← ∞
      pred(v) ← NULL
```

Say edge $u \to v$ is <u>tense</u> if $dist(u) + w(u \to v) < dist(v)$

```
RELAX(u→v):
  dist(v) ← dist(u) + w(u→v)
  pred(v) ← u
```

```
FORDSSSP(s):
  INITSSSP(s)
  while there is at least one tense edge
      RELAX any tense edge
```

Terminates with shortest paths & distances iff no negative cycle is reachable from s.

Lemma: In any instance of Ford SSSP, at any time, for any vertex v, value dist(v) is either ∞ or the length of a walk from s to v ending with (pred(v) → v).

Proof: (Using induction on # relaxations)

Last change to $dist(v)$ came from relaxing some edge $u \to v$.

We set $dist(v) \leftarrow dist(u) + w(u \to v)$.

& $pred(v) \leftarrow u$.

By induction $dist(u)$ was length of some s-to-u walk W.

Adding $u \to v$ to W, we get a walk from s to v of length $dist(u) + w(u \to v)$ ending with $u \to v$. ✓

$\Rightarrow$

If we set dist(v) to actual distance, pred(v) $\rightarrow$ v is the last edge on shortest path.

So we'll focus on correct dist values only.

Directed Acyclic Graphs:

- no (negative weight) cycles

(For now), let $dist(v)$ denote the true distance to v from s.

$$dist(v) = \begin{cases} 0 & \text{if } v = s \\ \min\limits_{u \to v} \left(dist(u) + w(u \to v) \right) & \text{o.w.} \end{cases}$$

eval in topological order!

```
DagSSSP(s):
  for all vertices v in topological order
      if v = s
          dist(v) ← 0
      else
          dist(v) ← ∞
          for all edges u→v
              if dist(v) > dist(u) + w(u→v)        ⟨⟨if u→v is tense⟩⟩
                  dist(v) ← dist(u) + w(u→v)       ⟨⟨relax u→v⟩⟩
```

$O(V+E)$ time

```
DagSSSP(s):
  InitSSSP(s)
  for all vertices v in topological order
      for all edges u→v
          if u→v is tense
              Relax(u→v)
```

```
PushDagSSSP(s):
  InitSSSP(s)
  for all vertices u in topological order
      for all outgoing edges u→v
          if u→v is tense
              Relax(u→v)
```

Always Works: Bellman-Ford

```
BELLMANFORD(s)
  INITSSSP(s)
  while there is at least one tense edge
      for every edge u→v
          if u→v is tense
              RELAX(u→v)
```

Let $dist_{\leq i}(v)$ denote the length of a shortest walk in G from s to v that uses $\leq i$ edges.

($dist_{\leq 0}(s) = 0$, $dist_{\leq 0}(v) = \infty$ for all $v \neq s$)

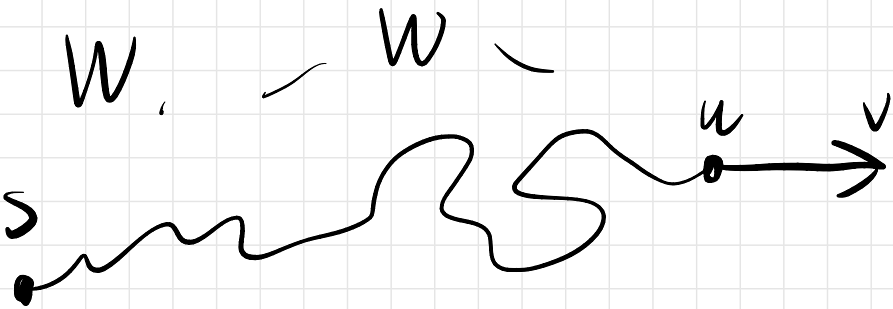
Lemma: For every vertex v & non-negative integer i, after i iterations of the while loop, we have $dist(v) \le dist_{\le i}(v)$

Proof: Lemma holds for $i = 0$.

Let W be shortest walk from s to v with $\le i$ edges. By definition W has length $dist_{\le i}(v)$.

If W has no edges, $v = s$ & $dist_{\le i}(v) = 0$. $dist(s) \le 0 = dist_{\le i}(v)$

O.w. let $u \to v$ be last edge of W.



After $i-1$ iterations $dist(u) \le dist_{\le i-1}(u)$. ✓

In ith iteration, we looked $u \to v$.

Either $dist(v) \le dist(u) + w(u \to v)$

or $u \to v$ was tense, so we set ✓

$dist(v) \leftarrow dist(u) + w(u \to v)$.

Either way,

$$\begin{aligned} dist(v) &\le dist(u) + w(u \to v) \\ &\le dist_{\le i-1}(u) + w(u \to v) \\ &= dist_{\le i}(v). \end{aligned}$$

Lemma still true with negative cycles.

But if no negative cycles...

shortest paths have $\le \underline{|V|-1}$ edges...

$dist_{\le |V|-1}(v) \le$ distance to v...

can stop after $|V|-1$ iterations

Iterations take $O(E)$ time.

$O(VE)$ time (if no neg. cycles)

Otherwise, still some tense edge after $|V|-1$ iterations

```
BellmanFord(s)
  InitSSSP(s)
  repeat V − 1 times
      for every edge u→v
          if u→v is tense
              Relax(u→v)
  for every edge u→v
      if u→v is tense
          return "Negative cycle!"
```

$O(VE)$ time