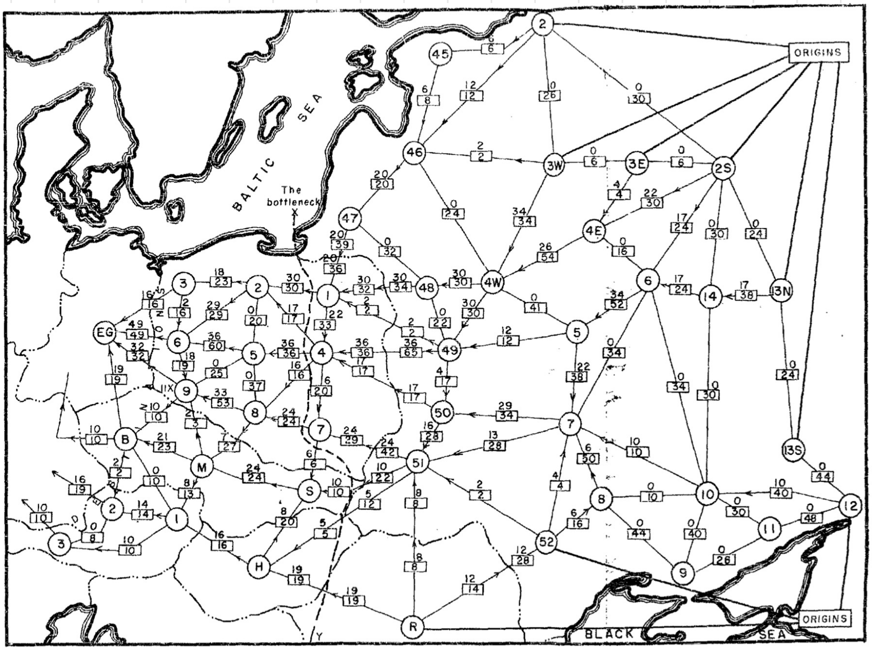


Fig. 7 — Traffic pattern: entire network available

Legend:

—·· — International boundary

(8) Railway operating division

$\overset{9}{\boxed{12}}$ Capacity: 12 each way per day. Required flow of 9 per day toward destinations (in direction of arrow) with equivalent number of returning trains in opposite direction

All capacities in $\frac{\text{trains}}{\sqrt{1000\text{'s of tons}}}$ } each way per day

Origins: Divisions 2, 3W, 3E, 2S, 13N, 13S, 12, 52 (USSR), and Roumania

Destinations: Divisions 3, 6, 9 (Poland); B (Czechoslovakia); and 2, 3 (Austria)

Alternative destinations: Germany or East Germany

Note IIX at Division 9, Poland

maximum flow &

minimum cut

Given a directed graph
$G = (V, E)$ & two vertices

s: source
t: target or sink

An (s,t)-flow is a
function $f: E \to \mathbb{R}_{\geq 0}$ that
satisfies the conservation
constraints for every
vertex v except for s & t:

$$\sum_{w} f(v \to w) = \sum_{u} f(u \to v)$$

(flow in = flow out)

$f(u \to v)$ is assumed $= 0$ if $u \to v \notin E$

Let $\delta f(v) := \sum_w f(v \to w) - \sum_u f(u \to v)$

(net flow out)

$\delta f(v) = 0$ for all $v \neq s, t$

The value of flow f is

$$|f| := \delta f(s) = \sum_w f(s \to w) - \sum_u f(u \to s)$$

$\Rightarrow \delta f(t) = -\delta f(s)$

$\left(0 = \sum_v \delta f(v) = \delta f(s) + \delta f(t)\right)$

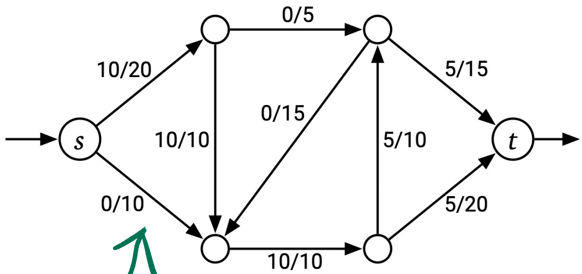
Given capacity function

$c : E \to \mathbb{R}_{\geq 0}$.

f is feasible if $f(e) \leq c(e)$ $\forall e \in E$

f <u>saturates</u> edge e if $f(e) = c(e)$

f <u>avoids</u> edge e if $f(e) = 0$.



flow / capacity

value = 10

maximum flow problem:

Given G, s, t, & c. Find a flow f of maximum value.

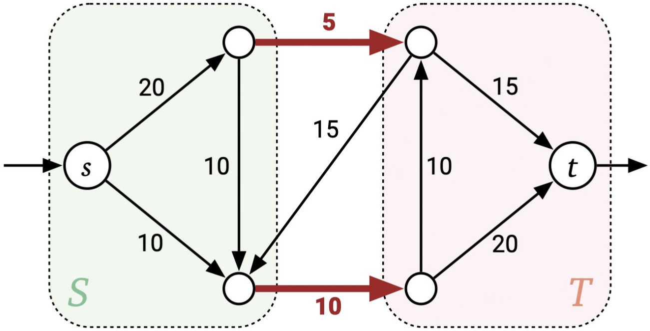
Minimum Cut

An **(s,t)-cut** is a partition of vertices into disjoint subsets S & T ($S \cup T = V$ & $S \cap T = \emptyset$) such that $s \in S$ & $t \in T$.

The **capacity** of cut (S,T) is the sum of capacities for edges starting in S & ending in T.

$$\|S,T\| = \sum_{v \in S} \sum_{w \in T} c(v \to w)$$

(say $c(u \to v) = 0$ if $u \to v \notin E$)



$\|S, T\| = 15$

minimum cut problem:

find an (s,t)-cut of min capacity

Lemma: The value of <u>any</u> feasible (s,t)-flow f is at most the capacity of <u>any</u> (s,t)-cut (S,T).

$$|f| = \partial f(s)$$
$$= \sum_{v\in S} \partial f(v)$$
$$= \sum_{v\in S}\sum_{w} f(v\to w) - \sum_{v\in S}\sum_{u} f(u\to v)$$
$$= \sum_{v\in S}\sum_{w\in T} f(v\to w) - \sum_{v\in S}\sum_{u\in T} f(u\to v)$$
$$\le \sum_{v\in S}\sum_{w\in T} f(v\to w) \qquad [f(e)\ge 0]$$
$$\le \sum_{v\in S}\sum_{w\in T} c(v\to w) \qquad [f(e)\le c(e)]$$
$$= \|S,T\|$$

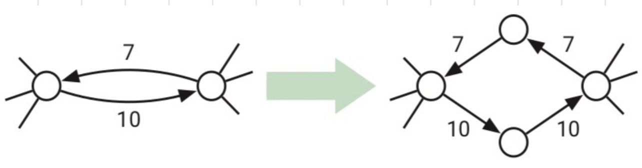
Are equal iff you avoid every T to S edge & saturate every S to T edge.

$\Rightarrow$ You have a max flow & a min cut.

Max flow Mincut Theorem
[Ford-Fulkerson '54]
([Elias, Feinstein, & Shannon '56]

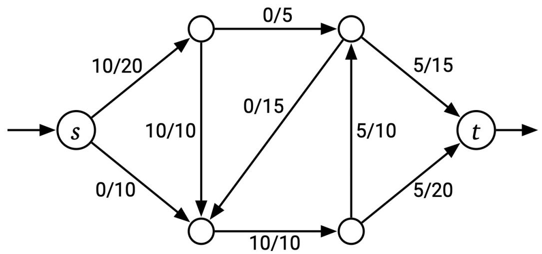
In any flow network, the value of the maximum (s,t)-flow equals the capacity of the min (s,t)-cut.

Assume graph is <u>reduced</u>
you don't have an edge $u \to v$ & its reversal $v \to u$ in E.



Consider any feasible flow f.

We'll try to increase value of f.



Sometimes you can't find a path of not-saturated edges. :(

The residual capacity

$c_f : V \times V \to \mathbb{R}$

$$c_f(u \to v) = \begin{cases} c(u \to v) - f(u \to v) & \text{if } u \to v \in E \\ f(v \to u) & \text{if } v \to u \in E \\ 0 & \text{o.w.} \end{cases}$$

more?

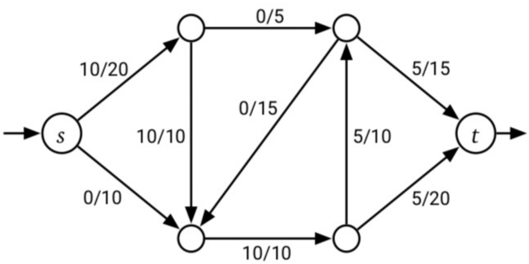
less?

non-negative

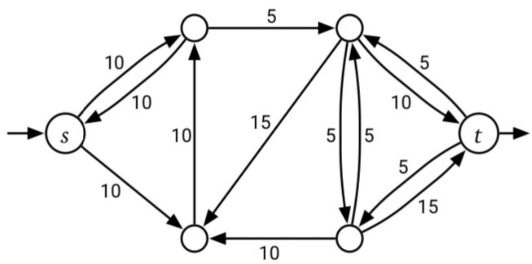
$c(u \to v)$ may be > 0 even if $u \to v \notin E$

residual graph $G_f = (V, E_f)$.

E_f is all edges with positive (>0) residual capacity.



$G, c, \& f$



$G_f \;\&\; c_f$

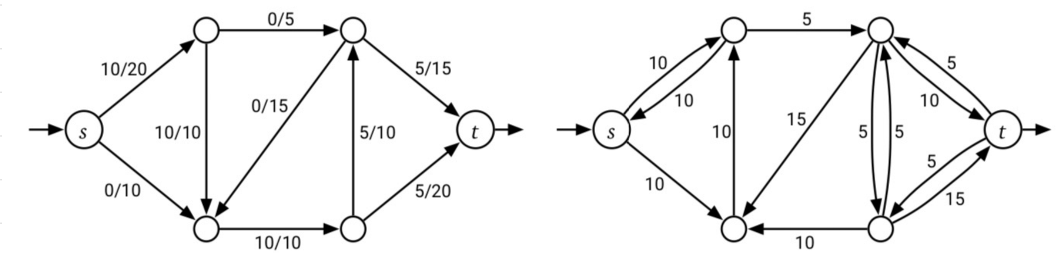
Suppose G_f has a path from s to t. Call it P.

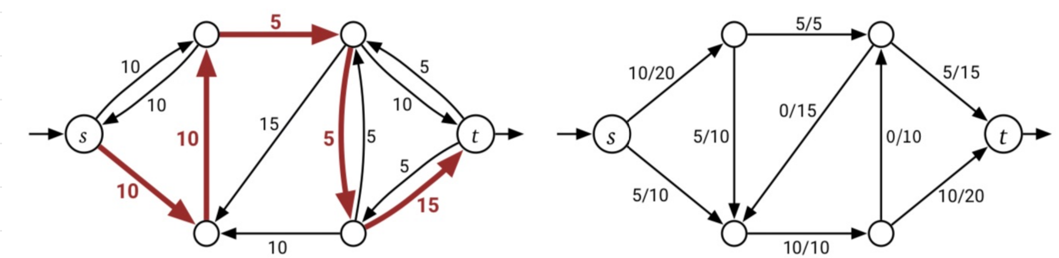
P is called an <u>augmenting path.</u>

Let $F := \min_{u \to v \in P} c_f(u \to v)$

We "push" F units of flow along P to make flow $f' : E \to \mathbb{R}$

$$f'(u \to v) = \begin{cases} f(u \to v) + F & \text{if } u \to v \in P \\ f(u \to v) - F & \text{if } v \to u \in P \\ f(u \to v) & \text{o.w.} \end{cases}$$





Push only changes $\partial f(s)$ + $\partial(t)$.

Increases $\partial f(s)$ by F.

So f' is still a flow,

Flow f' is feasible...

For any $u \to v \in E$

If $u \to v \in P$

$$
\begin{aligned}
f'(u \to v) &= f(u \to v) + F \\
&> f(u \to v) \\
&\geq 0
\end{aligned}
$$

$$
\begin{aligned}
f'(u \to v) &= f(u \to v) + F \\
&\leq f(u \to v) + c_f(u \to v) \\
&= f(u \to v) + c(u \to v) \\
&\quad - f(u \to v) \\
&= c(u \to v)
\end{aligned}
$$

(σ $0 \leq f(u \to v) \leq c(u \to v)$ if $v \to u \in P$)

$F > 0$, so $|f'| > |f| \Rightarrow f$ was not max

Otherwise, we'll see an (s,t)-cut (S,T) where $\|S,T\| = |f|$.