Given graph $G = (V, E)$

& edge capacities

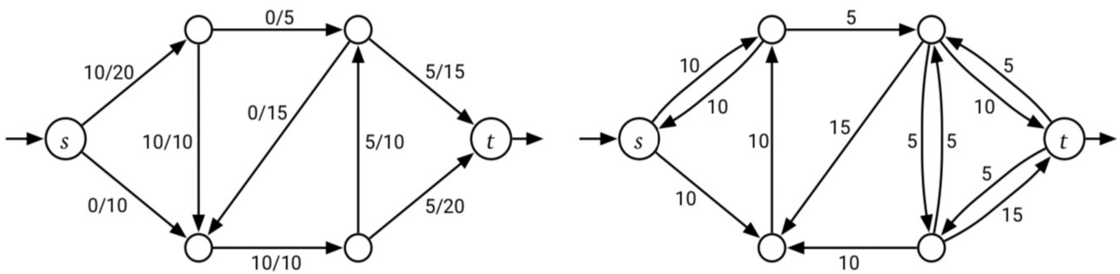
$c: E \to \mathbb{R}_{\geq 0}$.

Thm: Value of max

flow = capacity of

min cut.

Start with any feasible flow $f : E \to \mathbb{R}_{\geq 0}$



$f(e)/c(e)$

$|f| = 10$

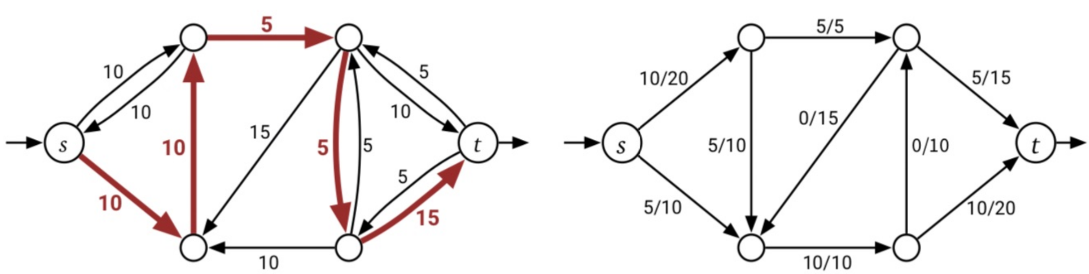
Residual capacities

$$C_f(u \to v) = \begin{cases} c(u \to v) - f(u \to v) & \text{if } u \to v \in E \\ f(v \to u) & \text{if } v \to u \in E \\ 0 & \text{o.w.} \end{cases}$$

Residual graph $G_f = (V, E_f)$

E_f those with > 0 residual capacity

If G_f has a path P from s to t, create a new flow f' by "pushing" flow along P.



Push min residual capacity for edges on P.

Value of flow increases by that much.

$\Rightarrow f$ was not maximum

If no path from s to t...

Let S be vertices in G_f you can reach from s.

Let $T = V \setminus S$.

$s \in S$

$t \notin S$, so (S, T) is an (s,t)-cut

For all $u \in S$, $v \in T$.

If $u \to v \in E$, then

$0 = c_f(u \to v) = c(u \to v) - f(u \to v)$

it's saturated.

If $v \to u$ in E, then

$0 = c_f(u \to v) = f(v \to v)$

it's avoided

So f is max value + (S, T) is min capacity with $|f| = ||S, T||$.

Ford-Fulkerson Augmenting Path Algorithm:

Start with a all-zero flow f. ($f(e) = 0 \; \forall e \in E$)

Repeatedly build residual graph + push flow along an augmenting path.

When you can't anymore, flow is maximum.

Analysis with integer capacities.

- initial flow is all integers (0)

- if we assume each flow f is all integers, residual capacities are integers

- so we always push a positive integer amount of flow

- so new flow is all integers

We push ≥ 1 unit of flow each iteration

Let f^* be the max value flow.

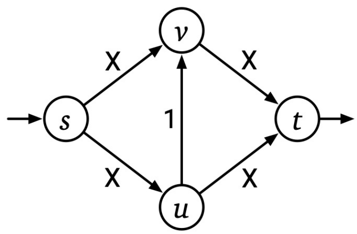
There are $\leq |f^*|$ iterations.

$O(E)$ time per iteration (build res. graph + search)

so... $O(E|f^*|)$ time total

18*) may not be polynomial in input size

may be exponential in # bits given as input



Could do X iterations if you always push on u→v or v→u!

"pseudo-polynomial time" - polynomial in terms of input <u>numbers</u> but not the input size (# bits)

$|S^*|$ could be small depending on application

Algorithm may never terminate if given real-valued capacities.

Could imply infinite loops if from roundind floating point values!

Edmonds-Karp: "Fat pipes"

Choose the augmenting path with largest bottleneck capacity (maximize amount pushed)

$O(E \log V)$ per iteration

Amount you have left to push decreases by a $1/|E|$ fraction each iteration.

$\Rightarrow |E| \cdot \ln |f^*|$ iterations

(with integer capacities)

$O(E^2 \log V \log |f^*|)$

time

"weakly polynomial"

Edmonds-Karp 2:

Choose augmenting path with smallest # edges.

$O(E)$ time per iteration (use BFS in G_f)

Let f_i the flow after i iterations.

Let $G_i = G_{f_i}$.

So $G = G_0$.

Let $\text{level}_i(v)$ be unweighted shortest path distance to v in G_i.

Lemma: $\text{level}_i(v) \geq \text{level}_{i-1}(v)$ for all v & i.

Proof: $\text{level}_i(s) = 0 = \text{level}_{i-1}(s)$ ✓

Otherwise, let $s \to \dots \to u \to v$ be shortest path to v in G_i.

$\text{level}_i(v) = \text{level}_i(u) + 1$.

By induction on level_i,

$\text{level}_i(u) \geq \text{level}_{i-1}(u)$

If $u \to v$ is in G_{i-1},

$$\text{level}_{i-1}(u) + 1 \geq \text{level}_{i-1}(v)$$

Otherwise, we must have push along $v \to u$. So shortest path to u used $v \to u$, so

$$\text{level}_{i-1}(u) + 1 > \text{level}_{i-1}(u) - 1$$
$$= \text{level}_{i-1}(v)$$

Either way,

$$\text{level}_i(v) = \text{level}_i(u) + 1$$
$$\geq \text{level}_{i-1}(u) + 1$$
$$\geq \text{level}_{i-1}(v)$$

Lemma: Any edge $u \to v$ disappears from residual graph $\leq |V|/2$ times

Proof: Suppose $u \to v$ in G_i & G_{j+1} but not in

$G_{i+1} \ldots G_j$.

$u \to v$ in ith augmenting path so $\text{level}_i(v) = \text{level}_i(u) + 1$

$v \to u$ in jth augmenting path so $\text{level}_j(u) = \text{level}_j(v) + 1$

So

$$\begin{aligned} \text{level}_j(u) &= \text{level}_j(v) + 1 \\ &\geq \text{level}_i(v) + 1 \\ &= \text{level}_i(u) + 2 \end{aligned}$$

So distance from s to u increased by 2.

Max path length is $|V|-1$

So $u \to v$ leaves $\leq |V|/2$ times.

So $\le |E||V|/2$ iterations,

$O(E^2V)$ time

(any ≥ 0 real capacities)

"strongly polynomial time"

Dinitz (70s): $O(EV^2)$ with almost same algorithm

⋮

Orlin (2012): $O(VE)$ time

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