CS 6301.002 Homework 3

Due Friday March 6th on eLearning

February 21, 2020

Please answer the following **3** questions, some of which have multiple parts. If asked for an $O(T(n))$ time algorithm, you may give a *randomized* algorithm with *expected* time $O(T(n))$.

1. **(From Mount)** This is a query variant of Problem 2(b) from Homework 2. You are given two sets $P = \{p_1, \ldots, p_m\}$ and $Q = \{q_1, \ldots, q_n\}$, which you may assume lie in the positive x, y-quadrant. The cannon is located at the origin, and its projectile function is restricted so that $a = 0$. In other words, given b and c, the projectile travels along the curve $y = bx - cx^2$. The objective is to preprocess the sets P and Q so that the following queries can be answered efficiently. Given a pair (b, c), determine whether the projectile passes above all the points of P and below all the points of Q.

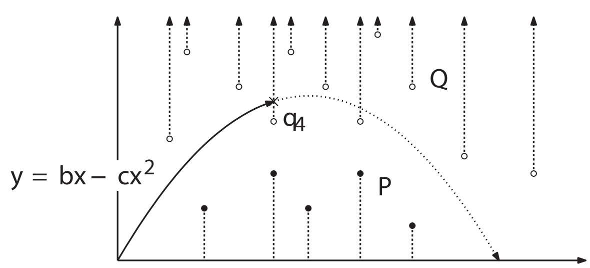

Figure 1. The example scenario for Problem 1.

Let $N = m + n$ be the total input size. The preprocessing time for your data structure should be at most $O(N \log N)$, your data structure should use space $O(N)$, and queries should be answered in time $O(\log N)$. Partial credit will be given if the space or preprocessing time is a bit higher, but the query time must be $O(\log N)$.

2. **(From Mount)** Given a set of n points in the plane, we define a subdivision of the plane into rectangular regions by the following rule. We assume that all the points are contained within a bounding rectangle. Imagine that the points are sorted in increasing order of y-coordinate. For each point in this order, shoot a bullet to the left, to the right, and up until each bullet hits an existing segment, and then add these three bullet-path segments to the subdivision (see Figure 2(a)).

   (a) Show that the resulting subdivision has size $O(n)$ (including vertices, edges, and faces).

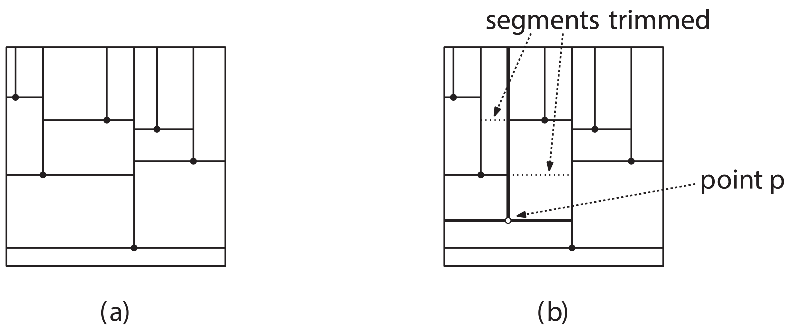

Figure 2. The example scenario for Problem 2.

(b) Consider the following algorithm that adds a new point p to the subdivision and restores the proper subdivision structure. We determine the rectangle containing p and subdivide it by shooting bullets to the left and right of p. Then, we iteratively following a bullet going up from p to the top of the bounding rectangle, trimming back each horizontal segment crossed by the bullet-path (see Figure 2(b)). Prove that if the n points are added in random order in this way, then the expected number of structural changes to the subdivision with each insertion is $O(1)$.

3. **(From Mount)** This problem arises from computational biology. You are given a protein molecule P that consists of n atoms. Each atom is represented by a circular disk in the plane of radius r (see Figure 3(b)). Let $P = \{p_1, \ldots, p_n\}$ denote the centers of these atoms.

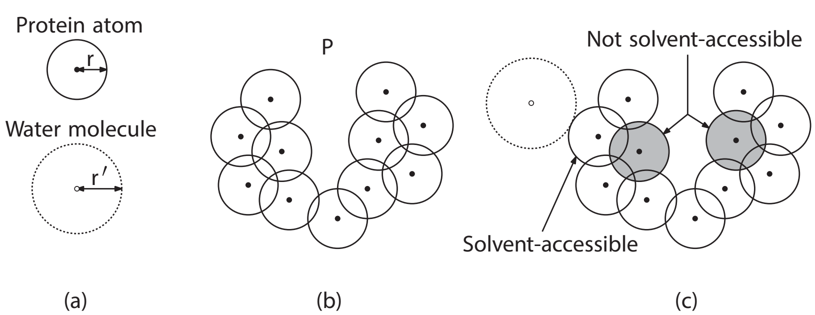

Figure 3. The example scenario for Problem 3.

The protein molecule lives in a solution of water. A water molecule is represented by a circular disk of radius $r' > r$ (see Figure 3(a)). Water molecule and protein molecules *cannot* overlap. We say that an atom of P is ***solvent-accessible*** if there exists a placement of a water molecule that touches this atom, but does *not* intersect any of the other atoms of P. (In Figure 3(c), the two shaded atoms are the only ones that are *not* solvent-accessible.)

(a) Consider the Voronoi diagram of the center points of the atoms of P. Show that an atom $p_i \in P$ is solvent-accessible if and only if there exists a point within p_i's Voronoi cell that is at distance at least $r + r'$ from p_i.

(b) Using (a), show that it is possible to determine all the atoms that are solvent-accessible in $O(n \log n)$ time.