CS 6301.002.20S Lecture 24—April 16, 2020

Main topics are #well_separated_pair_decompositions.

Well Separated Pair Decomposition (WSPD)

- Today, we're going to discuss some applications of well separated pair decompositions (WSPDs)
- Let's start with a review. Given a separation parameter $s > 0$, point sets A and B are s-well separated if A and B can be enclosed in sphere of radius r that are distance at least sr apart.
- An s-well separated pair decomposition (s-WSPD) of point set P is a collection of pairs of subsets $\{\{A_1, B_1\}, \{A_2, B_2\}, \ldots, \{A_m, B_m\}\}$ such that
  1. $A_i, B_i \subseteq P$ for all $1 \leq i \leq m$
  2. $A_i \cap B_i = \emptyset$ for all $1 \leq i \leq m$
  3. $\bigcup_{i=1}^m A_i \otimes B_i = P \otimes P$
  4. A_i and B_i are s-well separated for all $1 \leq i \leq m$

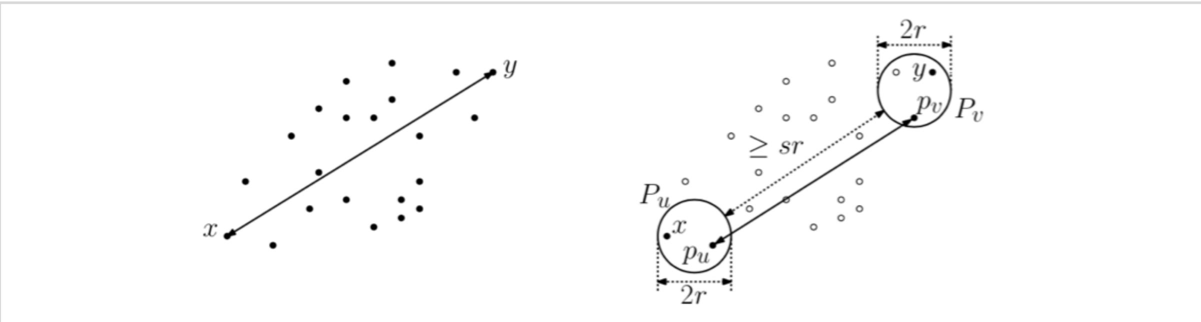
  where $A \otimes B$ is the set of unordered pairs from A and B.
- Last time, we saw how (for any $s \geq 2$), there exists an s-WSPD of size $O(s^d n)$ which can be constructed in $O(n \log n + s^d n)$ time.
- The WSPD can be represented as a set of unordered pairs of nodes from a compressed quadtree of P.
- For any node u, we'll let P_u be the points in u's cell and let rep(u) denote an arbitrary *representative point* in from P_u. We can compute these in representatives in $O(n)$ time given the compressed quadtree.
- Lemma: (WSPD Utility Lemma) If the pair $\{P_u, P_v\}$ is s-well separated and $x, x' \in P_u$ and $y, y' \in P_v$ then:

  i. $||x x'|| \leq 2/s * ||x y||$

  ii. $||x' y'|| \leq (1 + 4 / s) ||x y||$
- In other words, points within a subset are much closer than points between subsets.
- Proof:
  - We can enclose P_u and P_v in balls of radius r that are sr distance apart.
  - Therefore, $||x x'|| \leq 2r = (2r / sr) * sr \leq (2r / sr) * ||x y|| = 2 / s * ||x y||$.
  - And between triangle inequality and claim i., $||x' y'|| \leq ||x' x|| + ||x y|| + ||y y'|| \leq 2 / s * ||x y|| + ||x y|| + 2 / s * ||x y|| = (1 + 4 / s) ||x y||$.
- Now we can look at some applications.

Approximating the Diameter

- The *diameter* of a point set is the maximum distance between any pair of points in the set.
- We could compute it exactly in O(n^2) time by trying all pairs of points, and there's an O(n log n) time algorithm for points in the plane, but let's find a fast (1 + eps)-approximation algorithm for point sets in any constant dimensional Euclidean space.
- Given eps, let s = 4/eps and construct an s-WSPD.
- Let p_u = rep(u) and p_v = rep(v) for any pair of quadtree nodes u and v.
- For every well-separated pair {P_u, P_v}, compute ||p_u p_v|| and output the largest distance computed.
- There are O(s^d n) distances computed, so the whole thing takes O(n log n + s^d n) = O(n log n + n / eps^d), which is O(n log n) if eps is a constant.
- To prove correctness, let x and y be the points realizing the diameter and let {P_u, P_v} be the well-separated pair containing x and y respectively.
- By the Utility Lemma, ||x y|| ≤ (1 + 4 / s) ||p_u p_v|| = (1 + eps) ||p_u p_v||.
- {x, y} is the diametrical pair, so ||x y|| / (1 + eps) ≤ ||p_u p_v|| ≤ ||x y||. We have a (1 + eps)-approximation.

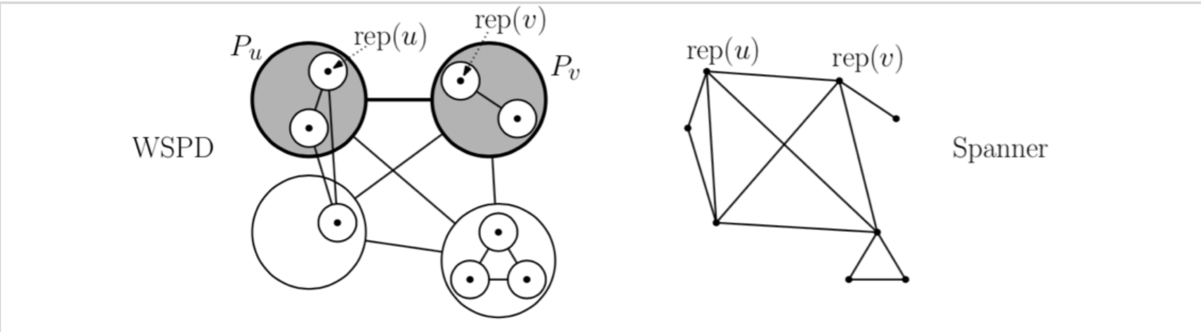


Closest Pair

- We can try the same algorithm for solving the *closest pair* problem: for every well-separated pair {P_u, P_v}, compute ||p_u p_v|| and output the smallest distance computed.
- The surprising thing is that this algorithm actually find the *exact* closest pair as long as s is large enough.
- Say {x, y} is the closest pair and that s > 2. Again let {P_u, P_v} be the well-separated pair containing x and y respectively.
- P_u and P_v lie in balls of radius r at distance at least sr > 2r apart, so ||p_u x|| ≤ 2r < sr ≤ ||x y||.
- If p_u ≠ x, then this contradicts x and y being the closest pair. p_v = y for the same reason. So the representatives we tested must actually be the closest pair!
- We can set s arbitrarily close to 2 so the running time of the algorithm is O(n log n + 2^d n) = O(n log n), assuming d is a constant.

Spanner Graphs

- Recall we can express all pairwise distances using between points in P using the Euclidean graph, the complete graph with edge weights equal to the distance between its endpoints.
- Unfortunately, it is a *dense* graph with Theta(n^2) edges. It would be nice to find a *sparse* graph with far fewer edges.
- Given a *stretch factor* t ≥ 1, a subgraph G of the Euclidean graph is called a *t-spanner* if for any pair of points x, y in P we have ||x y|| ≤ delta_G(x, y) ≤ t * ||x y|| where \delta_G(x , y) is the shortest path distance between x and y in G.
- I claimed in an earlier lecture that the Delaunay triangulation is a t-spanner for some 1.5846 ≤ t ≤ 2.418. This observation does not generalize to higher dimensions, and maybe we want better approximations of distance anyway.
- So here's what we'll do. Pick some s ≥ 2. We'll make a more concrete choice later.
- Compute an s-WSPD, and for each well-separated pair {P_u, P_v}, with representatives p_u = rep(u) and p_v = rep(v), add edge p_u p_v to the graph.
- G has O(s^d n) edges and takes O(n log n + s^d) time to construct.



- But is it a spanner? We need to prove for any x, y in P, ||x y|| ≤ delta_G(x, y) ≤ t * ||x y||.
- The first inequality is true, because G is a subgraph of the Euclidean graph.
- We'll prove the second inequality by induction on the Euclidean distance between two points.
- First, if x and y are joined by an edge in G, then \delta_G(x, y) = ||x y|| ≤ t * ||x y||.
- Now suppose otherwise. Again let {P_u, P_v} be the well-separated pair containing x and y respectively.
- By the triangle inequality,

$$\begin{aligned}\delta_G(x,y) &\leq \delta_G(x,p_u)+\delta_G(p_u,p_v)+\delta_G(p_v,y)\\ &\leq \delta_G(x,p_u)+\|p_up_v\|+\delta_G(p_v,y).\end{aligned}$$

- By the Utility Lemma, max(||x p_u||, ||p_v y||) ≤ 2 / s * ||x y|| ≤ ||x y||, and ||p_u p_v|| ≤ (1 + 4 / s) ||x y||.
- We can apply induction to say

$$\delta_G(x,y) \;\leq\; t\left(2\cdot\frac{2}{s}\cdot\|xy\|\right)+\left(1+\frac{4}{s}\right)\|xy\| \;=\; \left(1+\frac{4(t+1)}{s}\right)\|xy\|.$$

- So now to make the inequality work out, we just need 1 + 4(t + 1) / s ≤ t. So, set s := 4(t + 1) / (t - 1).

- Now

$$\delta_G(x,y) \ \leq \ \left(1 + \frac{4(t+1)}{4(t+1)/(t-1)}\right)\|xy\| \ = \ (1+(t-1))\|xy\| \ = \ t \cdot \|xy\|,$$

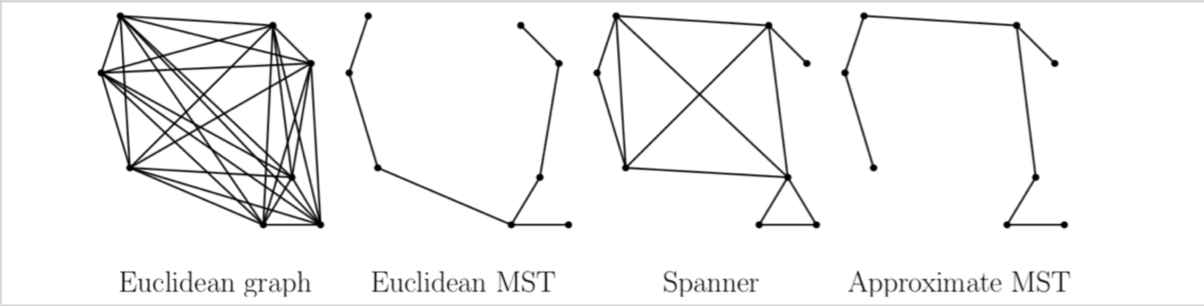
- Spanners are most interesting for small stretch factors, so let's assume t = 1 + eps for some 0 < eps ≤ 1.
- The size of the spanner is

$$O(s^d n) \ = \ O\left(\left(4\frac{(1+\varepsilon)+1}{(1+\varepsilon)-1}\right)^d n\right) \ \leq \ O\left(\left(\frac{12}{\varepsilon}\right)^d n\right) \ = \ O\left(\frac{n}{\varepsilon^d}\right).$$

- And it takes O(n log n + n / eps^d) time to build the thing.

Euclidean MST

- Let's finish up with a fast approximation algorithm for Euclidean Minimum Spanning Tree (MST).
- Computing the MST directly from the Euclidean graph takes Theta(n^2) time.
- Earlier, we discussed how the Delaunay triangulation actually contains the MST, giving us an O(n log n) time exact algorithm for the plane. What we'll do now works as an approximation for any constant dimension.
- First, we construct a (1 + eps)-spanner G = (V, E) using the algorithm we just discussed.
- Then, we compute the MST of G in O(V log V + E) time using a variant of Prim's algorithm with Fibonacci heaps. The total running time is O(n log n + n / eps^d).
- To see why it works, let w(x, y) = ||x y||. For any subgraph H of the Euclidean graph, let w(H) be the total weight of its edges. Finally, let pi_G(x, y) denote the shortest path from x to y in G so that w(pi_G(x, y)) = delta_G(x, y) ≤ (1 + eps) ||x y||.
- Let T be the minimum spanning tree. Form G' subset G by taking the union of edges of pi_G(x, y) for all xy in T. In other words, each edge of T is replaced by its shortest path in the spanner.
- G' must be connected, but it may not be a tree.
- We have w(G')
  - = sum_{xy in T} w(pi_G(x, y))
  - ≤ sum_{xy in T} (1 + eps) ||x y||
  - = (1 + eps) sum_{xy in T} ||x y||
  - = (1 + eps) w(T)
- On the other hand, we have less options when building the MST of G' than the MST of G, so the MST of G must weigh less than the MST of G'.
- We conclude w(MST(G)) ≤ w(MST(G')) ≤ w(G') ≤ (1 + eps) w(T).


Euclidean graph
Euclidean MST
Spanner
Approximate MST