CS 6301.002.20S Lecture 26—April 23, 2020

Main topics are #eps-nets and #set_cover.

Samples and Nets Review

- Let's start with a review of eps-samples and nets.
- For simplicity, say we have a range space (P, R) where P is finite.
- Given Q in R, Q's *measure* is the fraction of P that it contains. We denote it as mu(Q) = |Q intersect P| / |P|.
- Given a *sample* S subset P, the *estimate* of Q is mubar(Q) = |Q intersect S| / |S|.
- Given eps > 0, we'll say S is an eps-sample if for *any* range Q in R, we have |mu(Q) - mubar(Q)| ≤ eps.
- Given eps > 0, we'll say S is an eps-net if for any range Q in R with mu(Q) ≥ eps, Q contains at least one point of S. In other words, the net catches every large range Q.
- Theorem: Let (X, R) be a range space of constant VC-dimension, and let P be any finite subset of X. A random sample S subset P of size Omega((1 / eps^2) log (1 / eps)) is an eps-sample with constant probability.
- Theorem: A random sample S subset P of size Omega((1 / eps) log (1 / eps)) is an eps-net with constant probability.
- These theorems also work with non-negatively weighted point sets where measures and estimates are defined using the fraction of *total weight* for points in each subset compared to the weight of the whole set.
- For the weighted versions, sample points are selected with probability proportional to their weight.
- eps-samples have more obvious and direct use cases. Say you want to do *approximate range counting*. You want to know approximately how many people live within a certain distance of each major city in Texas. You only need to figure out locations for an eps-sample of the population.
- Applications for eps-nets are less obvious, and that's what I want to discuss today.

Sampling from a Disk

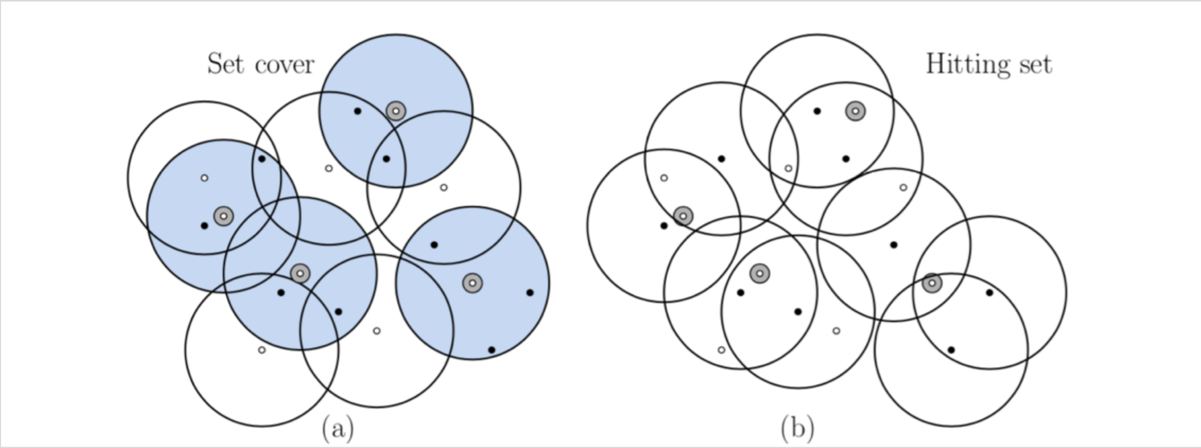
- Say we have a very large point set X in the plane and there's this unknown disk D_unknown. We can check if a point of X is in the disk, but that's it. We want to make a guess on what D_unknown looks like.
- In fact, let's settle for being wrong on only an eps fraction of the points.
- Pick an eps>0 and sample O(1/eps log(1/eps)) points at random from X. Test all the sample points, and find a disk enclosing only the samples that passed your test. Let that

disk be D.

- Disks have finite VC-dimension, and you can argue that the symmetric difference of two disks does as well.
- Our sample is an eps-net for X wrt the symmetric difference of disks with constant probability.
- If more than an eps-fraction of X lied in exactly one of D or D_unknown, then we would have sampled a point in that symmetric difference, contradicting D enclosing exactly the samples that passed the test.

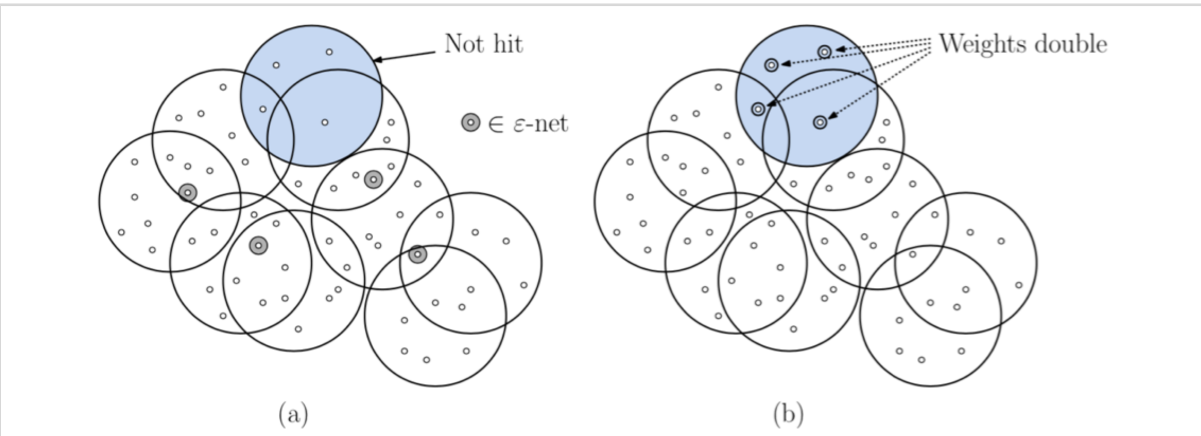
Approximation Algorithm for Set Cover

- The second application will be an approximation algorithm for the set cover problem.
- In the set cover problem, you are given an n-element ground set X and a collection of subsets R over X. (X, R) is really just a range set.
- The goal is to find a minimum size collection of subsets from R whose union is X.
- The problem is NP-hard, but there's a pretty simple greedy algorithm that does well: iteratively take a single set that includes as many uncovered elements as possible.
- This algorithm is an O(log n)-approximation, meaning you'll find a set cover with at most O(log n) times as many subsets as is optimal.
- In general, you cannot do better! No polynomial time algorithm can provide better than an Omega(log n)-approximation for every instance of set cover unless P = NP.
- However, if (X, R) has constant VC-dimension, then you can get an approximation ratio of O(log OPT) where OPT is the size of the smallest set cover. That could be a lot less if OPT << n!
- I'll focus on the simpler case we saw last time of trying to place routers to cover a set of sites.
- We have an m-element point set P in R^2 (we'll see why I wrote m shortly) and an n-element point set T with represents possible locations of transmission towers.
- We'll make things even easier and focus on the decision version of set cover: given a value k, can we pick k points from T such that every point from P is in the disk of radius 1 centered at T?
- We'll make an algorithm that finds a set of size O(k log k) if such a set exists. It may report failure otherwise.
- To make things a bit easier, we'll actually look at a "dual" problem called *hitting set*. A point p in P lies in a disk of radius 1 centered at t in T if and only if t lies in a disk of radius 1 centered at p.
- So, we'll try to find a set of O(k log k) points from T (which has size n) that hit all m disks centered at points of P.



Iterative Reweighting

- So, here's the algorithm for hitting set: again, we have a set T of n points and a set P of m points. We want to find a subset of T that hits all the unit disks centered at points of P.
- Given k, our algorithm will find a hitting set of size O(k log k) if a hitting set of size k exists.
- We assign each point from T a weight initially set to 1.
- We're going to compute eps-nets on T where the range space is the set of unit disks. Again, an eps-net has at least one point in each disk that contains at least an eps fraction of the *total weight*.
- Here are the details:
  1. Let eps = 1 / (4k). Let k' = c k log k for a suitably large constant c.
  2. Compute the weighted eps-net N of T of size k'.
     - Again, we randomly sample k' points where a point is sampled with probability proportional to its weight. Keep trying until you finally do get an eps net.
  3. Check if there is a disk that it not hit by N. If so, double the weight of each point of T in that one disk and return to step 2.
     - lg means log_2
     - If you've done more than 2k lg (n /k) iterations, output failure; I claim there is no hitting set of size k.
  4. If you reach this step, N must be a hitting set of size k' = O(k log k). Output N.



- So the algorithm is polynomial time (with high probability). You'll almost certainly find an eps-sample within log n attempts in each iteration, and there are a polynomial number of iterations in the worst case.
- Now we need to argue that the algorithm will find a hitting set within $2k \lg (n / k)$ iterations if one of size k exists.
- Here's the high level idea behind the argument before we get into the math.
- If N is not a hitting set, then we find that one disk that is not hit by N. But that set has at most an eps fraction of the total weight. Doubling the weight of that set increases the total weight by a (1 + eps) factor at most.
- On the other hand, the optimal hitting set does hit that disk we missed. So we'll double that weight of at least one of k points in the optimal hitting set.
- The weight of the optimal hitting set eventually starts increasing at a faster rate than the weight of the whole point set. The hitting set is part of the point set, though, so we can only repeat the procedure a bounded number of times.
- Now let's formalize the idea.
- Assume there is some optimal hitting set H of size k.
- Let W_i denote the total weight of all points in T after the ith iteration.
- $W_0 = n$, because all points start with weight 1.
- If iteration i does not return a hitting set, then there is some disk not hit by the eps-net N, meaning it has total weight at most eps W_{i-1}. We double the weight of points in this disk, so
    - $W_i \leq W_{i-1} + \text{eps } W_{i-1} = (1 + \text{eps})W_{i-1}$.
- Because $W_0 = n$, $W_i \leq (1 + \text{eps})^i n \leq n * e^{\text{eps } i}$ (for any x, $1 + x \leq e^x$).
- Optimal hitting set H hits every disk including that disk we missed, so at least one of its k points had its weight doubled.
- For $1 \leq j \leq k$, let $t_i(j)$ denote the total number of times the jth optimal point has had its weight doubled by the end of iteration i.
- Let $W_i(H)$ be the total weight of H after the ith iteration. We have
    - $W_i(H) = \sum_{j=1}^k 2^{t_i(j)}$.
- At least one of these k points has its weight doubled each iteration, so $\sum_{j=1}^k t_i(j) \geq i$.
- Because 2^x is convex, $W_i(H)$ is minimized when those k $t_i(j)$'s are as equal as possible subject to that bound on their sum. In other words,
    - $W_i(H) \geq \sum_{j = 1}^k 2^{i / k} = k 2^{i / k}$.
- Finally, $W_i(H) \leq W_i$, because $H \subset T$.
- So, $k 2^{i / k} \leq n e^{\text{eps } i}$.
- Taking the lg of both sides, we have $\lg k + i / k$
    - $\leq \lg n + \text{eps } i \lg e$
    - $= \lg n + (i / 4k) \lg e$

    - $\le$ lg n + i / 2k (because lg e ≅ 1.45 < 2).
- A bit of algebra shows i / 2k $\le$ lg n - lg k = lg (n / k), so i $\le$ 2k lg(n / k).
- Meaning we cannot have more than 2k lg(n / k) iterations if there is a hitting set H of size k.