CS 6301.008.18S Lecture—February 22, 2017

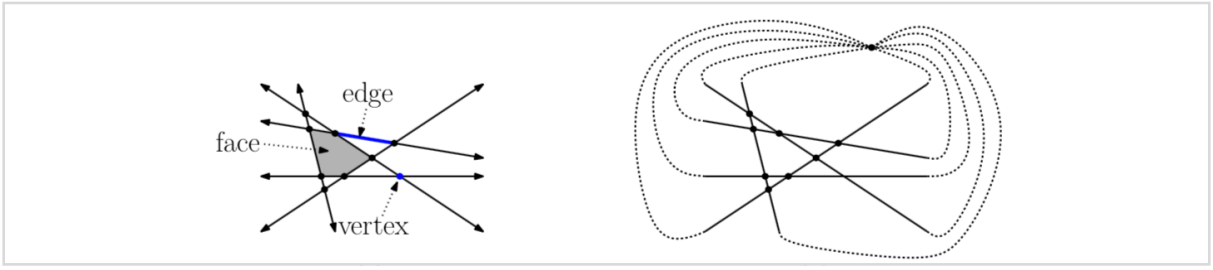
Main topics are #line_arrangements and #zone_theorem.

Prelude

- Homework 2 was due today. Please submit on eLearning ASAP if you haven't already.

Line Arrangements

- Let L be a finite set of lines in the plane.
- The lines subdivide the plane into a *cell complex* or planar subdivision called the *arrangement* of L, denote A(L).
- Intersection points are the vertices, segments between intersection points form the edges, and and polygonal regions between lines form faces.
- Like the Voronoi diagram, there are unbounded faces. And like the Voronoi diagram, the easiest way to deal with them is to add a vertex point at infinity and make it the endpoint of all the unbounded edges.
- After doing that, you get a proper embedded planar graph. You can store it as a DCEL.



- Today, we'll discuss some basic combinatorial properties of line arrangements and how to construct them efficiently. Tuesday, I'll show you a couple applications of line arrangements which take advantage of point-line duality like we saw earlier in the semester.

Combinatorial Properties

- The *combinatorial complexity* of an arrangement is the total number of vertices, edges, and faces.
- The arrangement is *simple* if no three lines intersect at a common point, which is guaranteed by our normal general position assumption for lines.
- Let's also assume no two lines are parallel.
- With those two assumptions, we can prove the following exact bounds. They only get smaller if the assumptions aren't true.
- Lemma: Let A(L) be a simple arrangement of n lines L in the plane. Then:

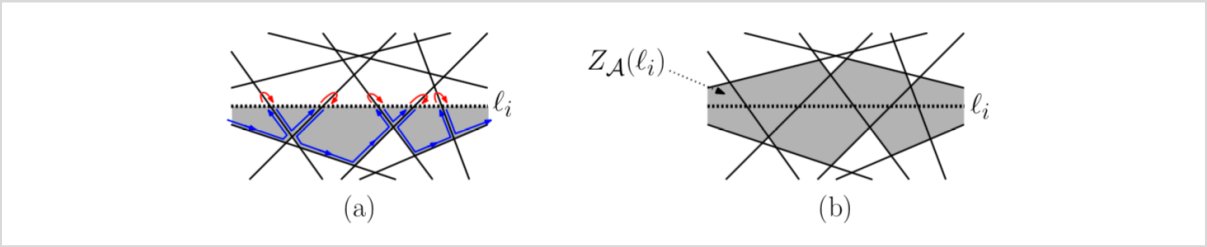
    - there are (n choose 2) vertices
    - there are n^2 edges
    - there are (n choose 2) + n + 1 faces
- Proof:
    - Every pair of lines intersects at exactly one point.
    - By induction:
        - One line consists of one unbounded edge.
        - Given n lines, remove one of them to get an arrangement with (n-1)^2 edges.
        - Putting that line back splits the others into n - 1 new edges, and the new line is split into n new edges total.
        - (n-1)^2 + (n-1) + n = n^2.
    - By Euler's formula for planar graphs, v - e + f = 2 where v, e, and f are the number of vertices, edges, and faces, respectively.
        - v = (n choose 2) + 1 and e = n^2.
        - f = 2 - v + e = 2 - (1 - (n choose 2)) + n^2 = 2 - (1 + n(n-1) / 2) + n^2 = 1 + n^2 / 2 + n / 2 = 1 + n(n-1) / 2 + n = (n choose 2) + n + 1.
- As an aside, similar properties hold in higher dimensions. There, we would consider arrangements of hyperplanes that make a polyhedral cell complex where d hyperplanes make a vertex, d - 1 make an edge, d - 2 make a face, and so on. The combinatorial complexity is Theta(n^d).

Incremental Construction

- If we're going to use line arrangements, we should probably figure out how to construct them.
- We'll use another incremental algorithm. However, this one is *not* randomized.
- The main reason we get away with a deterministic algorithm is that we can afford an O(n^2) time construction, since the arrangement has that complexity.
- Let L = {ell_1, …, ell_n}. We'll add each line ell_i in O(i) time for a total running time of O(n^2).
- Let L_i = {ell_1, …, ell_i} and A(L_i) be the arrangement of the first i lines.
- Say it's time to insert line ell_i. We first find the leftmost (unbounded) face of A(L_{i-1}) containing the line. To do that, observe that the lines at x = -infinity are sorted top to bottom by increasing order of their slopes. We just compare the slope of ell_i to all others in O(i) time to find where it falls in the order.
- ell_i cuts through a sequence of i - 1 edges from the other lines. We need to figure out which ones they are. Once we do so, we can split them and update the DCEL in O(1) time per edge we cut.
- The surprising thing is that we don't need to do anything particularly clever to find the

edges we're going to split.

- What we'll do is iteratively walk along each of the faces that our new line cuts through. We can walk around a face in O(1) time per edge using the DCEL.
  - Say we figure out where ell_i enters an edge on the left side of a face.
  - We walk counterclockwise along the face's edges until we find the other edge that intersects ell_i.
  - Then we jump to the other side of that edge to walk along the next face.
  - See (a) below.

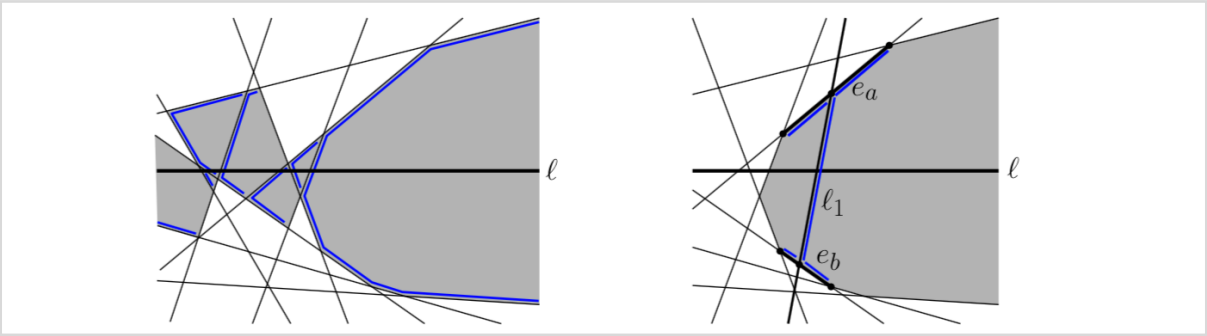


- Everything that isn't the walk takes O(i) time. So how long do we spend walking along the faces?
- Naively, we split i - 1 edges, so we pass through i faces.
- Each face is bounded by at most i lines, so a single face traversal takes O(i) time.
- But together that implies O(i^2) time for all the face traversals. We need a better argument.

Zone Theorem

- Let L be a set of n lines and A be their arrangement.
- Let ell be any line *outside* of L. The *zone* of ell in A, denoted Z_A(ell), is the set of faces in A intersected by ell. See (b) above.
- If we get a good bound on the total complexity/number of edges in the zone for a the line ell_i, then we learn how long those walks take.
- Zone Theorem: The total number of edges in all faces of the zone Z_A(ell) is at most 6n.
- So, if we're inserting ell_i into an arrangement of i - 1 lines, we only walk around O(i) edges.
- Proof:
  - For simplicity, let's rotate the plane so ell is horizontal.
  - We'll assume none of the n lines are parallel to ell.
  - Split the edges into two groups. First are *left bounding* edges for which an incident zone face lies in their *right* halfplane (so they bound the left side of the face). There's also *right bounding* edges.
  - We'll prove there are at most 3n left bounding edges and 3n right bounding edges
    - Edges crossed by ell are both left and right bounding, so we're overcounting by a bit.

- We'll proceed using induction.
- For n = 1, there's exactly one left bounding edge (the whole line of L) and 1 ≤ 3 = 3n.
- For higher n, consider the rightmost line of A(L) to intersection ell. Call it ell_1. Let L' be the other n - 1 lines of L, and let A' be their arrangement.



- Inductively, there are at most 3(n - 1) left bounding edges in Z_{A'}(ell).
- But what if we add back ell_1?
- ell_1 intersects ell within the rightmost face of Z_{A'}(ell).
- All the edges of the rightmost face are left bounding edges.
- Faces are convex, so ell_1 intersects the face at exactly two edges e_a and e_b.
- ell_1 contains a brand new left bounding edge, and it splits e_a and e_b into two left bounding edges each for a net increase in 3 left bounding edges.
- Are there any other new left bounding edges?
- Any left bounding edges from ell_1 lying above e_a lie in the region bounded by ell_1 and e_a's line. But that whole region lies above the zone.
- You can say a similar thing for trying to find new edges below e_b.

Uses and a Caveat

- Building an arrangement like this is useful for a variety of applications, some of which also use point-line duality.
- For some of these examples, you need to remove the general position assumptions on the lines, but the details aren't too hard. Again, the textbook has the details. Here are a few example problems you can solve easily in O(n^2) time using line arrangements, without going into details.
  - General position test: Given a set of n points in the plane, determine whether any are collinear.
  - Minimum area triangle: Given a set of n points in the plane, determine the minimum area triangle with vertices selected from the points.
  - Visibility graph: Given line segments in the plane, two points are *visible* if the interior of the line segment joining them intersects none of the segments. Given n non-intersecting line segments, compute the *visibility graph* which has endpoints for vertices and an edge between pairs of endpoints that are visible to one another.

    - Maximum stabbing line: Given n line segments in the plane, compute the line ell that stabs the maximum number of line segments.
    - Ham sandwich cut: Given n red points and m blue points, find a single line ell that simultaneously bisects both point sets.
        - This is always possible, no matter how the points are arranged.
        - In fact, given d sets of colored points in R^d, we can use a single d- 1 dimensional hyperplanes to bisect every color set.
        - In other words, we can cut a sandwich with bread, ham, and cheese in half with a single chop no matter how the ingredients are arranged.
- Unfortunately, the arrangement based algorithms also require O(n^2) space to actually store the arrangement, which is fine for something like visibility graph that has that output size anyway. Not so great a general position test.
- Next Tuesday, we'll briefly discuss a strategy to avoid the space usage for some of these problems at the cost of an extra O(log n) in running time.
- We'll also go over some applications of planar arrangements and point-line duality in more detail.