CS 6301.008.18S Lecture—March 1, 2017

Main topics are #Delaunay_triangulations, #3D_convex_hulls, #Voronoi-diagrams, and #3D_upper_envelopes.

Prelude

- Project proposals are due Tuesday. Feel free to email me or visit office hours if you want to discuss ideas.

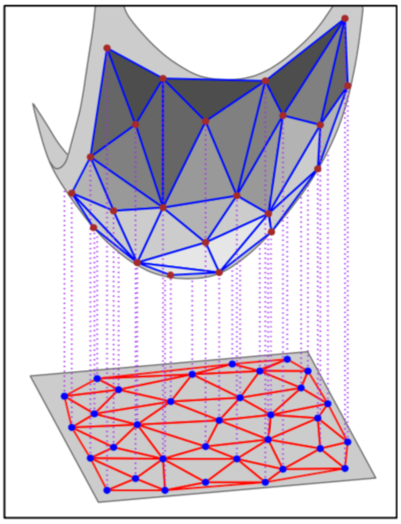
From Planar Subdivisions to 3D Polytopes

- So far, we have mostly focused on algorithms and data structures for the plane.
- Some of these problems and their algorithms generalize pretty easily:
  - Linear programming
  - kd-trees and orthogonal range trees
- And for others, the definitions generalize, but the algorithms don't generalize so cleanly:
  - Convex hulls: Intersection of all halfspaces containing all points
  - Upper/lower envelopes: Intersection of upper / lower halfspaces for a given set of (hyper)planes
  - Voronoi diagrams
  - Delaunay triangulations
- One informal explanation is that objects in higher dimensions are in some sense strictly more complicated than those in lower dimensions.
- In particular, we can sometimes solve complicated lower dimensional problems by transforming them to seemingly easier problems in higher dimensions.
- I want to give two examples today:
  - The Delaunay triangulation of a planar point set is topologically equivalent to the boundary complexity of a convex hull of a 3D point set.
  - The Voronoi diagram of a planar point set is topologically equivalent to the boundary complex of the intersection of a bunch 3D halfspaces.
- This also gives us an opportunity to look at some neat 3D stuff that isn't yet another example of incremental construction.

Delaunay Triangulations and Convex Hulls

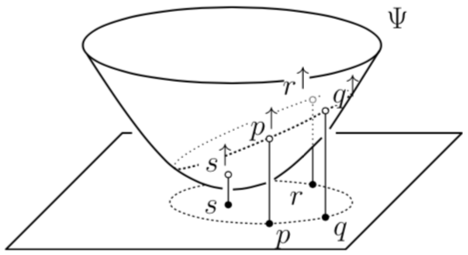
- Both examples rely heavily on a certain paraboloid in 3D Psi with equation $z = x^2 + y^2$.
- To relate this paraboloid to points in the plane, we'll define *vertical projections*, sometimes called *lifts*. The vertical projection of $p = (p_x, p_y)$ is a 3D point on Psi: $p^{\uparrow} = (p_x, p_y, p_x^2 + p_y^2)$.

- Since this map is a bijection, I might get sloppy and refer to p as the projection of p↑.
- Given P in the plane, let P↑ denote the projection of every point of P onto Psi.
- Now, consider the *lower convex hull* of P↑, the portion of the convex hull visible from z = -infinity.
- Assuming general position, each face is a triangle. But there's something special about these triangles…



- I claim, in fact, that if you project the edges of the lower convex hull back down to the plane, you get the Delaunay triangulation!
- In particular, given p, q, and r in P, triangle p↑q↑r↑ is a face of the lower convex hull of P↑ if and only if triangle pqr is a triangle of the Delaunay triangulation of P.
- To see that, we need to recall tests for if we have a Delaunay triangulation or convex hull:
  - Delaunay condition: p, q, and r form a Delaunay triangle if and only if no other point of P is in the circumcircle for triangle pqr.
  - Convex hull condition: p↑, q↑, and r↑ form a convex hull face if and only if no other point of P↑ is below the plane passing through p↑, q↑, and r↑.
- So to prove my claim, it suffices to prove the following lemma.
- Lemma: Let p, q, r, and s be distinct points of P. Point s lies within the circumcircle for triangle pqr if and only if s↑ lies beneath the plane passing through p↑, q↑, and r↑.
- To start, let's establish the general relationship between lower halfplanes in 3D and circles in 2D.
- Fix some non-vertical plane in 3D lying tangent to Psi at point (a, b, a^2 + b^2).
- Let's figure out the equation for this plane. To do that, observe partial z / partial x = 2x and partial z / partial y = 2y.
- These partial derivatives evaluate to 2a and 2b at the tangent point, so the plane has the form z = 2ax + 2by + gamma.
- Using the tangent point to solve for gamma, we have a^2 + b^2 = 2a * a + 2b * b + gamma, implying gamma = -(a^2 + b^2). The tangent plane is z = 2ax + 2by - (a^2 + b^2).

- Now, suppose we shift the plane upward by lambda^2 units. We get the plane z = 2ab + 2by - (a^2 + b^2) + lambda^2.
- The intersection points of this plane with Psi are ones in which x^2 + y^2 = 2ax + 2by - (a^2 + b^2) + lambda^2 which implies (x - a)^2 + (y - b)^2 = lambda^2.
- But that's the equation for a circle in the plane centered at (a, b).
- So what happened here? We could have created any plane via this process. So we have a proof that a plane's intersection with Psi is the projection of a circle in the plane.
- In particular, the parts of Psi *under* the plane are the projection of points *inside* the circle.



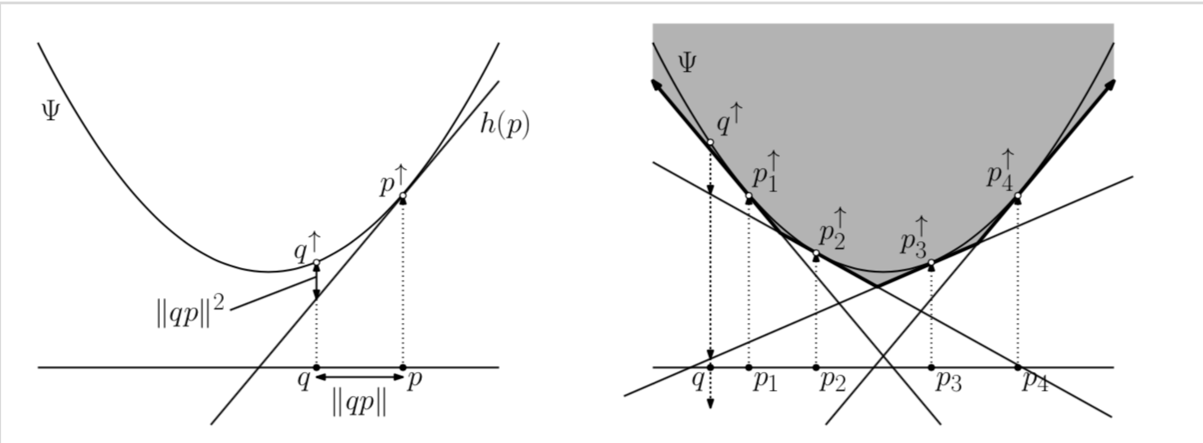
- So take p, q, and r from the lemma statement. The projection of their circumcircle is the intersection of the unique plane through p↑, q↑, and r↑ with Psi.
- So if s lies inside the circle, then s↑ lies on the part of Psi below the plane. Otherwise, it lies on the part of Psi outside the plane. We've finished proving the lemma.
- So, Theorem: Given p, q, and r in P, the triangle pqr is in the Delaunay triangulation if and only if the triangle triangle p↑q↑r↑ is a face of the lower convex hull of P↑.
    - The triangle is in the Delaunay triangulation if and only if there is no point s in P inside the circumcircle if and only if there is no point s↑ in P↑ below the plane through p↑q↑r↑ if and only if triangle p↑q↑r↑ is a face of the lower convex hull.
- Before we move on, I want to give one nice application of this result.
- Part of building the Delaunay triangulation is to test if a point s is inside the circumcircle of three points p, q, and r.
- Say p, q, and r are counterclockwise around the triangle. We can define a function inCircle(p, q, r, s) that is negative if s is outside the circle, 0 if s is on the circle, and positive otherwise, similar to those orientation tests we discussed in the first lecture.
- inCircle(p, q, r, s) can be defined as the result of an orientation test of the projected points so that orient(p↑, q↑, r↑, s↑) is negative if s is *above* the plane through p, q, and r, 0 if s is in the plane, and positive if s is below.

$$\text{orient}(p^\uparrow, q^\uparrow, r^\uparrow, s^\uparrow) = \text{inCircle}(p, q, r, s) = \text{sign}\det\begin{pmatrix} p_x & p_y & p_x^2 + p_y^2 & 1 \\ q_x & q_y & q_x^2 + q_y^2 & 1 \\ r_x & r_y & r_x^2 + r_y^2 & 1 \\ s_x & s_y & s_x^2 + s_y^2 & 1 \end{pmatrix}.$$

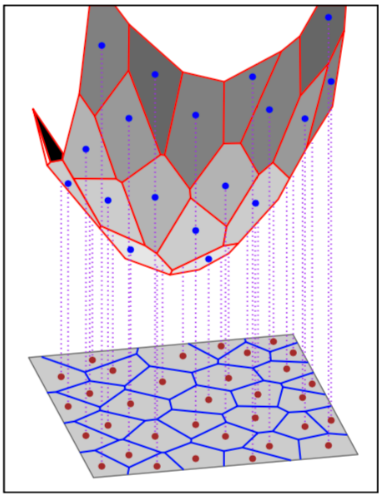
- So you still need to do some algebra for a formal proof this all works, but you might get some intuition at least.

Voronoi Diagrams and Upper Envelopes

- Earlier in the semester we discussed how lower convex hulls and upper envelopes are "dual" to one another by point-line duality.
- Later, we defined Delaunay triangulations as the planar graph dual of Voronoi diagrams.
- Since two wrongs make a right (or two duals make a primal), it makes some intuitive sense then that Voronoi diagrams are related to upper envelopes.
- Let's formalize that notion. We'll start by mapping a planar point set to Psi, not as points but instead planes.
- Earlier, we argued that for a point p = (a, b) in P, the plane tangent to Psi at p↑ is z = 2ax + 2by - (a^2 + b^2).
- Let h(p) denote this plane.
- Now, consider any other point q = (q_x, q_y) in R^2. Recall q↑ = (q_x, q_y, q_z) where q_z = q_x^2 + q_y^2.
- Because h(p) is tangent to Psi, it lies below every point of Psi, including q↑.
- The vertical distance from q↑ to h(p) is therefore:
  - q_z - (2aq_x + 2bq_y - (a^2 + b^2))
  - = (q_x^2 + q_y^2) - (2aq_x + 2bq_y - (a^2 + b^2))
  - = (q_x^2 - 2aq_x + a^2) + (q_y^2 - 2bq_y + b^2)
  - = (q_x - a)^2 + (q_y - b)^2
  - = ||qp||^2
- So the vertical distance from q↑ to h(p) is the squared distance from q to p.

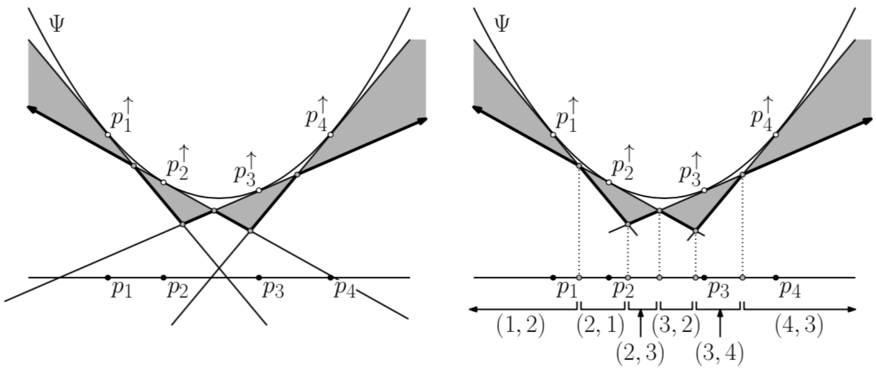


- Which implies…
- Lemma: Let H(P) = {h(p) : p in P}. For any point q in R^2, the vertical ray directed downward from q↑ intersects the planes of H(P) in order of increasing distances of points in P from q.
- So let U(P) be the upper envelope of H(P). If q↑ lies directly above h(p), then p is its closest point.
- Theorem: Let U(P) be the upper envelope of tangent hyperplanes to Psi at each point p↑ for p in P. The Voronoi diagram of P projects to the boundary complex of U(P).



Higher Order Voronoi Diagrams and Arrangements

- So far we discussed Voronoi diagrams based on the nearest site to each point.
- There's a generalization called the order-k Voronoi diagram: subdivide the plane based on the k nearest sites to each point.
- So for order-2 diagrams, each cell is labeled with a pair of sites indicating which are the two closest to points in the cell.
- Those hyperplanes H(P) defined earlier create an arrangement in R^3, dividing up space into polytopes, faces, edges, and vertices.
- I talked about levels on Tuesday. The kth level of the arrangement is the set of points in 3D for which there are at most k - 1 hyperplanes above and at most n - k hyperplanes below.
- From the earlier lemma on distances, the bottom boundary of the kth level is the projection of the order-k Voronoi diagram. The k hyperplanes below you tell you exactly which are the k nearest sites, and in what order.
- If all you care about is subsets of nearest sites and not their order, you actually have a refinement of that diagram.



- Finally, the lower envelope of H(P) tells you which sites are farthest away from each point. This is called the farthest-point Voronoi diagram.