CS 6301.008.18S Lecture—March 29, 2017

Main topics are #well_separated_pair_decompositions.

Well Separated Pair Decomposition (WSPD)

- Today, we're going to discuss some applications of well separated pair decompositions (WSPDs)
- Let's start with a review. Given a separation parameter s > 0, point sets A and B are s-well separated if A and B can be enclosed in sphere of radius r that are distance at least sr apart.
- An s-well separated pair decomposition (s-WSPD) of point set P is a collection of pairs of subsets {{A _ 1, B _ 1}, {A*2, B_2}, …, {A*m, B_m}} such that
  1. A _ i, B _ i subset P for all 1 ≤_ i ≤_ m
  2. A _ i intersect B _ i = emptyset for all 1 ≤_ i ≤_ m
  3. union _ {i=1}^m A _ i otimes B _ i = P otimes P
  4. A _ i and B _ i are s-well separated for all 1 ≤_ i ≤_ m

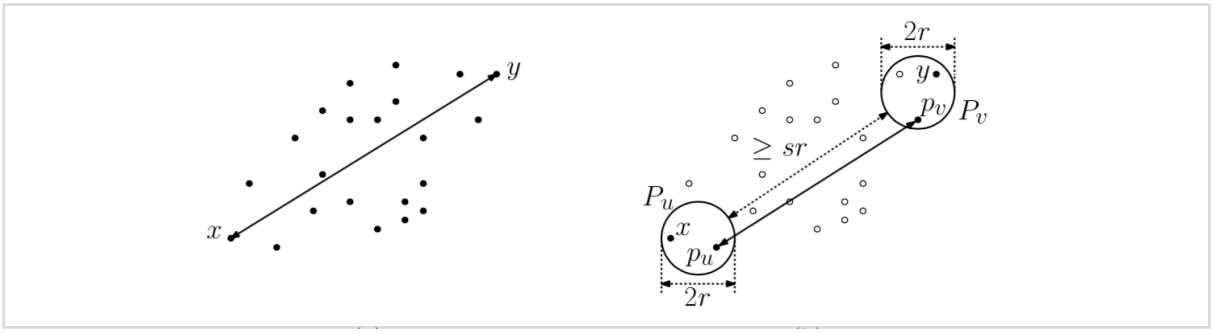
  where A otimes B is the set of unordered pairs from A and B.
- Last time, we saw how (for any s ≥ 2), there exists an s-WSPD of size O(s^d n) which can be constructed in O(n log n + s^d n) time.
- The WSPD can be represented as a set of unordered pairs of nodes from a compressed quadtree of P.
- For any node u, we'll let P_ u be the points in u's cell and let rep(u) denote an arbitrary / representative point/ in from P_ u. We can compute these in representatives in O(n) time given the compressed quadtree.
- Lemma: (WSPD Utility Lemma) If the pair {P_ u, P_ v} is s-well separated and x, x' in P_ u and y, y' in P_ v then:

  i. ||x x'|| ≤ 2/s * ||x y||

  ii. ||x' y'|| ≤ (1 + 4 / s) ||x y||
- In other words, points within a subset are much closer than points between subsets.
- Proof:
  - We can enclose P _ u and P _ v in balls of radius r that are sr distance apart.
  - Therefore, ||x x'|| ≤ 2r = (2r / sr) * sr ≤ (2r / sr) * ||x y|| = 2 / s * ||x y||.
  - And between triangle inequality and claim i., ||x' y'|| ≤ ||x' x|| + ||x y|| + ||y y'|| ≤ 2 / s * ||x y|| + ||x y|| + 2 / s * ||x y|| = (1 + 4 / s) ||x y||.
- Now we can look at some applications.

Approximating the Diameter

- The /diameter/ of a point set is the maximum distance between any pair of points in the set.
- We could compute it exactly in O(n^2) time by trying all pairs of points, and there's an O(n log n) time algorithm for the plane, but let's find a fast (1 + eps)-approximation algorithm for point sets in any constant dimensional Euclidean space.
- Given eps, let s = 4/eps and construct an s-WSPD.
- Let p*u = rep(u) and p*v = rep(v) for any pair of quadtree nodes u and v.
- For every well-separated pair {P*u, P*v}, compute ||p*u p*v|| and output the largest distance computed.
- There are O(s^d n) distances computed, so the whole thing takes O(n log n + s^d n) = O(n log n + n / eps^d), which is O(n log n) if eps is a constant.
- To prove correctness, let x and y be the points realizing the diameter and let {P*u, P*v} be the well-separated pair containing x and y respectively.
- By the Utility Lemma, ||x y|| ≤ (1 + 4 / s) ||p*u p*v|| = (1 + eps) ||p*u p*v||.
- {x, y} is the diametrical pair, so ||x y|| / (1 + eps) ≤ ||p*u p*v|| ≤ ||x y||. We have a (1 + eps)-approximation.

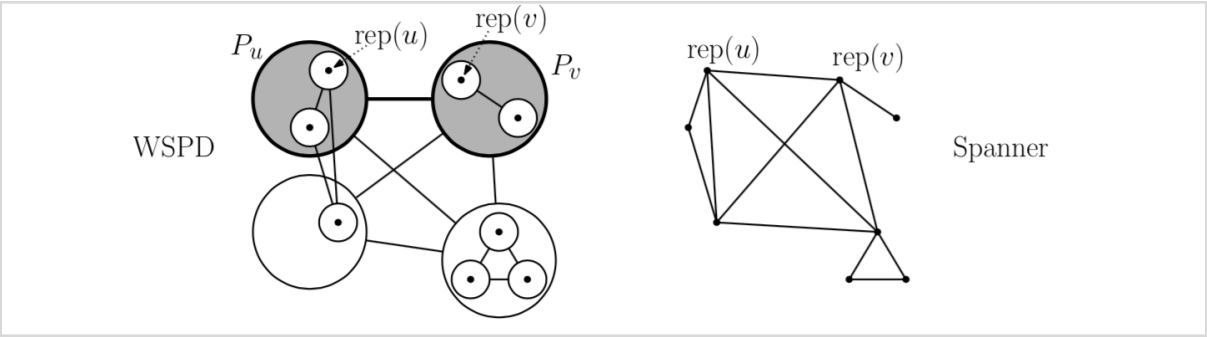


Closest Pair

- We can try the same algorithm for solving the /closest pair/ problem: for every well-separated pair {P*u, P*v}, compute ||p*u p*v|| and output the smallest distance computed.
- The surprising thing is that this algorithm actually find the /exact/ closest pair as long as s is large enough.
- Say {x, y} is the closest pair and that s > 2. Again let {P*u, P*v} be the well-separated pair containing x and y respectively.
- P*u and P*v lie in balls of radius r at distance at least sr > 2r apart, so ||p_u x|| ≤ 2r < sr ≤ ||x y||.
- If p*u ≠ x, then this contradicts x and y being the closest pair. p*v = y for the same reason. So the representatives we tested must actually be the closest pair!
- We can set s arbitrarily close to 2 so the running time of the algorithm is O(n log n + 2^d n) = O(n log n), assuming d is a constant.

Spanner Graphs

- Recall we can express all pairwise distances using between points in P using the Euclidean graph, the complete graph with edge weights equal to the distance between its endpoints.
- Unfortunately, it is a /dense/ graph with Theta(n^2) edges. It would be nice to find a /sparse/ graph with far fewer edges.
- Given a /stretch factor/ t ≥ 1, a subgraph G of the Euclidean graph is called a /t-spanner/ if for any pair of points x, y in P we have ||x y|| ≤ delta*G(x, y) ≤ t* * ||*x y*|| *where* *delta*G(x , y) is the shortest path distance between x and y in G.
- I claimed in an earlier lecture that the Delaunay triangulation is a t-spanner for some 1.5846 ≤ t ≤ 2.418. This observation does not generalize to higher dimensions, and maybe we want better approximations of distance anyway.
- So here's what we'll do. Pick some s ≥ 2. We'll make a more concrete choice later.
- Compute an s-WSPD, and for each well-separated pair {P*u, P*v}, with representatives p*u* = *rep(u) and p*v = rep(v), add edge p*u p*v to the graph.
- G has O(s^d n) edges and takes O(n log n + s^d) time to construct.



- But is it a spanner? We need to prove for any x, y in P, ||x y|| ≤ delta_G(x, y) ≤ t * ||x y||.
- The first inequality is true, because G is a subgraph of the Euclidean graph.
- We'll prove the second inequality by induction on the Euclidean distance between two points.
- First, if x and y are joined by an edge in G, then \delta_G(x, y) = ||x y|| ≤ t * ||x y||.
- Now suppose otherwise. Again let {P*u, P*v} be the well-separated pair containing x and y respectively.
- By the triangle inequality, \delta_G(x, y)
  - ≤ \delta*G(x, p*u) + \delta*G(p*u, p*v)* + *delta*G(p_v, y)
  - ≤ \delta*G(x, p*u) + ||p*u p*v|| + \delta*G(p*v, y)
- By the Utility Lemma, max(||x p*u||, ||p*v y||) ≤ 2 / s * ||x y|| ≤ ||x y||, and ||p*u p*v|| ≤ (1 + 4 / s) ||x y||.
- We can apply induction to say \delta_G(x, y)
  - ≤ t(||x p*u|| + ||p*v y||) + ||p*u p*v||
  - ≤ t(2 * 2 / s * ||x y||) + (1 + 4 / s) ||x y||

    - = (1 + 4(t + 1) / s) ||x y||.
- So now to make the inequality work out, we just need 1 + 4(t + 1) / s ≤ t. So, set s := 4(t + 1) / (t - 1).
- Now delta_G(x, y)
    - ≤ (1 + 4(t + 1) / (4(t + 1) / (t - 1))) ||x y||
    - = (1 + (t - 1)) ||x y||
    - = t * ||x y||.
- Spanners are most interesting for small stretch factors, so let's assume t = 1 + eps for some 0 < eps ≤ 1.
- The size of the spanner is O(s^d n)
    - = O((4((1 + eps) + 1) / ((1 + eps) - 1))^d n)
    - ≤ O((12 / eps)^d n)
    - = O(n / eps^d).
- And it takes O(n log n + n / eps^d) time to build the thing.

Euclidean MST

- Let's finish up with a fast approximation algorithm for Euclidean Minimum Spanning Tree (MST).
- Computing the MST directly from the Euclidean graph takes Theta(n^2) time.
- Earlier, we discussed how the Delaunay triangulation actually contains the MST, giving us an O(n log n) time algorithm for the plane. What we'll do now works in any constant dimension.
- First, we construct a (1 + eps)-spanner G using the algorithm we just discussed.
- Then, we compute the MST using any O(v log v + e) time algorithm like Prim's with Fibonacci heaps. The total running time is O(n log n + n / eps^d).
- To see why it works, let w(x, y) = ||x y||. For any subgraph H of the Euclidean graph, let w(H) be the total weight of its edges. Finally, let pi*G(x, y) denote the shortest path from x to y in G so that w(pi*G(x, y)) = delta_G(x, y) ≤ (1 + eps) ||x y||.
- Let T be the minimum spanning tree. Form G' subset G by taking the union of edges of pi_G(x, y) for all xy in T. In other words, each edge of T is replaced by its shortest path in the spanner.
- G' must be connected, but it may not be a tree.
- We have w(G')
    - = sum*{xy in T} w(pi*G(x, y))
    - ≤ sum_{xy in T} (1 + eps) ||x y||
    - = (1 + eps) sum_{xy in T} ||x y||
    - = (1 + eps) w(T)
- On the other hand, we have less options when building the MST of G' than the MST of G,

so the MST of G must weigh less than the MST of G′.

- We conclude w(MST(G)) ≤ w(MST(G′)) ≤ w(G′) ≤ (1 + eps) w(T).