A metric space \((X, d)\) where \(X\) is a possibly infinite set and \(d : \times \times \to \mathbb{R}_{\geq 0}\) is a distance function satisfying

\[
d (x, x) = 0 \quad \forall \ x \in X
\]
\[
d (x, y) > 0 \quad \forall \ x, y \in X \quad x \neq y
\]
\[
d (x, y) = d (y, x) \quad \forall \ x, y \in X
\]
\[
d (x, z) \leq d (x, y) + d (y, z)
\]
(triangle inequality) \(\forall \ x, y, z \in X\)
Ex: Euclidean distance in $\mathbb{R}^d$
- Hausdorff distance between point sets in $\mathbb{R}^d$
- Fréchet distance between curves in $\mathbb{R}^d$

$k$-center: given $(X, d) \cup k$, find $C \subseteq X$ such that $|C| = k$ to minimize $\max_{x \in X} \min_{c \in C} d(x, c)$.

- NP-hard
- Greedy $2$-approx
- But no poly time approx scheme!
Metric Embeddings
say $1 \times 1 = n$ ($n$) distances
want to map $(X, d)$ to another, simpler / better structured metric

Given metrics $(X, d_x)$ and $(Y, d_y)$, a function $f : X \rightarrow Y$ is a D-embedding if $\exists r > 0$ s.t.
$\forall x, x' \in X$
$r d_x (x, x') \leq d_y (f(x), f(x'))$
A 1-embedding scaling all distances by same factor $r$ called isometric embeddings.

Sometimes $D$ must be large for some $(X,d_X) + (Y,d_Y)$.

Graphic metric: given $G_x = (V_x, E_x)$

$X := V$

$d(u,v) = \#$ edges on shortest path

If $G_x$ is a cycle on $n$-vertices
If $G$ is any tree, any embedding is a $\Omega(n)$-embedding.
Say we have $n$ points in $\mathbb{R}^d$. Here, $d$ is pretty large.

Johnson-Lindenstrauss "Lemma":

Let $X$ be any $n$ points in $\mathbb{R}^d$ and let $0 < \delta < 1$. There exists a $(1 + \delta)$-embedding of $X$ into $\mathbb{R}^k$ where $k = O\left(\frac{\log n}{\delta^2}\right)$. 
Best to think of $X$ as vectors.

Will really show today, if you fix $0 < \varepsilon < 1 + \epsilon$ and let

$$k \geq \frac{4 \ln n}{\varepsilon^2 - \varepsilon^3 / 3},$$

there is an embedding into $\mathbb{R}^k$ s.t.

$$(1 - \varepsilon)\|u - v\|^2 \leq \|\Phi(u) - \Phi(v)\|^2 \leq (1 + \varepsilon)\|u - v\|^2 \quad \forall u, v \in X.$$

Embedding: Pick a good $k$-D

subspace of $\mathbb{R}^d$ and project all points into subspace.
So, if necessary, will show any arbitrary vector has square length of by $\leq (1+\varepsilon)$.

With a random subspace will succeed with prob. at least $1 - \frac{2}{n^2}$.

Prob. of failing with all pairwise distance vectors is $\leq \left(\begin{array}{c} n \\ 2 \end{array}\right) \cdot \frac{2}{n^2} < 1$. 
Can focus just on unit vectors.

To pick a random unit vector:

Let $x, \ldots, x_d$ be independent Gaussian random variables from $N(0,1)$. Let $Y = \frac{1}{\|x\|} <x, \ldots, x_d>$. 

random unit vector
(a point chosen uniformly at random from \( S_{d-1} \))

\[ Z \in \mathbb{R}^k \] projection of \( Y \) using first \( k \) coordinates only.

\[ L := \|Z\|^2 \]

\[ \mu := \mathbb{E}[L] \]

\[ \mathbb{E}[L] = \mathbb{E}\left[ \sum_{i=1}^{k} x_i^2 \right] \]

\[ \mathbb{E}\left[ \sum_{i=1}^{k} x_i^2 \right] = \frac{\mathbb{E}[x_1^2] + \cdots + \mathbb{E}[x_k^2]}{\mathbb{E}[x_1^2] + \cdots + \mathbb{E}[x_d^2]} \]

\[ = \frac{k}{d} \]
Lemma: Let $k \leq d, \beta \in \mathbb{R}_{\geq 0}$, 
\[ Pr [L \leq B, k/d] \leq (c^{k/2})(1 - \beta + \frac{1}{\ln B}) \]
\[ Pr [L \geq B, k/d] \leq \varepsilon \]

Back to original question...

Let $S$ be a random $kD$ subspace. Project each $v_i \in \mathbb{R}^d$

to get $v_i' \in \mathbb{R}^k$.

Take $v_i, v_j \in \mathbb{R}^d$

Let $L' = \| v_i' - v_j' \|^2$.
\[ m' = E[L'] = \frac{k}{d} \| v_{\omega} - v_{j} \|^2, \]

\[ Pr[L' \leq (1-\epsilon)m'] \leq \exp \left( \frac{k/2 (1-(1-\epsilon)^2)}{\ln(1-\epsilon)} \right) \]

\[ \leq \exp \left( -k \epsilon^2/4 \right) \]

\[ = \exp \left( -2 \ln n \right) \leq \frac{1}{n^2} \]

\[ Pr[L' \geq (1+\epsilon)m'] \leq \frac{1}{n^2} \]

Real map is \[ f(lv_j) = \sqrt{\frac{k}{n}} v_j. \]

Can almost guarantee a good embedding by scaling \( k \) by a
constant.
Bourgain's Theorem:

Let \((X, d)\) be any metric over \(n\) elements. There exists an \(O(\log n)\)-embedding of \(X\) into \(\mathbb{R}^{O(\log^2 n)}\).

**Algorithm:**

For \(i \leftarrow 1\) to \(c \log n\)

For \(j \leftarrow 1\) to \(\log n\)

Construct \(A_i \subseteq X\) by selecting points independently with prob. \(2^{-j}\)
Define $d(x, A_{ij}) := \min_{y \in A_{ij}} d(x, y)$

$f(x) := \max_{\hat{\omega}_j \leq \omega_j} \left\{ \frac{1}{\exp(\omega_j)} \right\} \text{ for } \omega_j \leq 0$.

$\{1 \leq j \leq \lceil \log n \rceil \}$. 