Polygon Triangulation

Dual graph: one vertex per triangle & edges between adjacent triangles' vertices.
Always a tree if you triangulate a simple polygon with no holes.
Two vertices are visible if the line segment between them stays interior to the polygon.

A diagonal is a line segment between visible vertices.

Any polygon with \( \geq 4 \) vertices has a pair of visible but non-adjacent vertices.

Their diagonal divides the polygon. Recursively partition polygon halves to get a triangulation.
Triangulations use $n-2$ triangles for $n$ vertices.

Can be many triangulations.
Fasten Alg in Two Steps

1) Decompose polygon into monotone pieces in $O(n \log n)$ time [Lee, Preparata '77]

2) Triangulate each monotone polygon in $O(n)$ time [Garey et al., '78]
Formal definitions:

**Polygonal curve**: sequence of line segments joined end-to-end

*Closed* if first endpoint = last

Segments = edges

Endpoints = vertices

*Simple* if nonincident elements don't intersect

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polyangular curve  
simple  
closed and simple
Closed + simple curves break plane into an interior + exterior.

Also called simple polygon.

A polygonal curve $C$ is monotone with respect to $L$ if every line orthogonal to $L$ intersects $C$ along one component.

Strictly monotone if intersections are 1 point.
Simple polygon $P$ is monotone wrt $l$ if we can split $P$'s boundary into two curves monotone wrt $l$.

horizontally or \underline{x-monotone}

is $l$ is the $x$-axis.

\begin{itemize}
  \item $x$-monotone polygon
  \item Splitting diagonals
  \item Monotone decomposition
\end{itemize}
Triangulating an \( x \)-monotone polygon:

Given \( x \)-monotone polygon with \( n \) vertices...

Vertices denoted \( v_1, \ldots, v_n \) in left-to-right order

(merge vertex chains for top & bottom of polygon to find sorted order in \( O(n) \) time)
Want to triangulate as much of left of line as we can...

A reflex vertex has interior angle $\geq \pi (180^\circ)$

Others are non-reflex.

Reflex chain: a sequence of reflex vertices along polygon boundary
Desired invariant: For $i \geq 2$, suppose we just processed $v_i$. The untriangulated region left of $v_i$ has two monochain chains. One is a reflex chain from $v_i$ to some vertex $u$. Other chain is a single edge from $u$ to some vertex right of $v_i$. 

Main invariant | Case 1 | Case 2(a) | Case 2(b)
---|---|---|---
$u$ | $v_{i-1}$ | $v_i$ | $v_i$ | $v_i$ | $v_i$ | $v_i$ |
$u$ | $v_{i-1}$ | $v_i$ | $v_{i-1}$ | $v_i$ | $v_i$ | $v_i$ |

new choice for $u$
Algorithm + Proof:

For $\hat{w} = 2$,

For $\hat{w} > 2$ suppose invariant holds through $v_{\hat{w}-1}$.

Case 1 ($v_s$ on opposite chain from $v_{\hat{w}}$): Add diagonals to all vertices $v_{\hat{w}-1}$ to (but not including) $v_s$. Now $u \leftarrow v_{\hat{w}-1}$.
Case 2 (\(v_i\) on same chain as \(v_{i-1}\))

2(a) (\(v_{i-1}\) is non-reflex):

At least one vertex before \(v_{j}\) is visible to \(v_i\). Add diagonal back to last vertex \(v_j\) of reflex chain visible to \(v_{i-1}\).

2(b) (\(v_{i-1}\) is reflex): Do nothing.

To implement: Store reflex chain in a stack & keep a flag saying if reflex chain is on top or bottom.
Analysis: Constant time per vertex we sweep + added diagonal so $O(n)$ overall.
Monotone Subdivision:

A polygon is not $x$-monotone iff it has a scan reflex vertex:

a reflex vertex where both incident edges go left or both go right

merge vertex

split vertex
Say sweep line is on
Split vertex \(v\)

\[ u = \text{helper}(e_a) \]

\(e_a\): edge immediately above \(v\)

\(\text{helper}(e_a)\): rightmost vertex \(u\) left of sweep line s.t. vertical segment from \(e_a\) to \(u\) is entirely in polygon
Can always add a diagonal from $v$ to $\text{helper}(e_a)$. 

sweep line

$e_a$

$e_b$

$\text{helper}(e_a)$

sweep line

$e_a$

$e_b$

$e_1$

$e_2$

$e_3$

$e_4$

$e_5$

$e_6$

$\text{helper}(e_3)$

$\text{helper}(e_5)$

$\text{helper}(e_1)$