Given $S = \{s_1, \ldots, s_n\}$, let $n$ line segments.

Segments could share an endpoint, but distinct endpoints have distinct x-coordinates.

1) Add a bounding box
2) Shoot "bullet path" up & down from each endpoint to first
They hit also called vertical extensions.

Decomposes plane into a trapezoidal map/decomposition.

Vertical sides are walls.
Top and bottom are line segments.
Lemma: \( \leq 6n + 4 \) vertices
\( \leq 3n + 1 \) trapezoids

Proof: 3 vertices per endpoint...
Any segment's endpoints lie on left wall of three trapezoids.
Leftmost trap has no endpoint on left wall.

Every trapezoid "depends on" \( \leq 4 \) segments.
Randomized incremental algorithm:

Start only with bounding box

\[ S_{\tilde{u}} : = \{ s_1, \ldots, s_3 \} \]

\[ T_{\tilde{u}} : \text{map for } S_{\tilde{u}} \]

Say we have \( T_{\tilde{u}-1} \). What happens when adding \( s_{\tilde{u}} \)?
1) Figure out which trapezoid contain lefthandpoint of $s_u$.
(can do this in $O(\log n)$ expected time)

2) Walk along $s_u$ left to right. Observe which trapezoids you pass.

3) Add extensions from $s_u$ & trim back walls we cross.
OLD time per new wall $\Rightarrow$
OLD time per new trapezoid.

So if $k$, new trapezoids
$O(k)$ time (ignoring point location)
Analysis: Permute segments before insertion to avoid \( \Theta(n^2) \) worst case time.

Lemma:
\[ E[k_w] = O(1) \]
(over all permutations)

\( S_w \), \( O(n) \) time total outside point location.

Proof: Fix some subset \( S_w \) of first \( w \) segments but not the insertion order.
Say trapezoid $\triangle$ depends on segment $s$ if adding $s$ last creates $\triangle$ in last iteration.

Trapezoids that depend on $s$ Segments that $\triangle$ depends on

Let $\delta(\triangle, s) = 1$ if $\triangle$ depends on $s$ and 0 otherwise.
\[ F \left( k, \omega \right) = \Xi \left( \text{prob. that } s \text{ is last} \right), \]
\[ \Xi \left( s \in S \left( \Delta, \omega \right) \right) \]
\[ = \frac{1}{2} \frac{1}{s \in S \left( \Delta, \omega \right)} \]
\[ \Xi \left( s \in S \left( \Delta, \omega \right) \right) \]
\[ \Xi \left( 4 \right) \]
\[ = \frac{1}{\Delta \epsilon} (3 \omega + 1) \cdot 4 \]
\[ = 0 \left( 1 \right) \]
Point Location

Given a planar subdivision preprocess to build a data structure.

Should take any query point \( q \) and quickly return \( q \)'s face.

We'll really do vertical segment ray-shooting queries. What lies immediately below \( q \)?
Use a rooted directed acyclic graph.

Each node has out degree 0: a leaf.

Has one source (in-degree 0) called the root.

1-1 correspondence between leaves and trapezoids.
**Internal nodes**: references on endpoint $p$ of a segment. Children correspond to query $q$ being left or right of $p$.

**$y$-nodes**: references a segment $s$. Is $q$ above or below $s$?

**Query**: Start at root, Go down based on comparisons, until you hit a leaf.

**Time prop.**: To length of path.
Suppose we add segment $s$...

Remove leaves for now—gone trapezoids & replace with small trees.

Identify leaves pointing to a common trapezoid.
Analysis:

Space is $O(n)$ (in expectation)