CS 6363.003 Homework 2

Due Sunday March 7th on eLearning

February 21, 2021

Please answer the following **4** questions, some of which have multiple parts.

1. (a) Suppose we are given two sorted arrays $A[1 .. n]$ and $B[1 .. n]$. Describe an algorithm to find the median element (the element of rank n) in the union of A and B in $O(\log n)$ time. You may assume that the arrays contain no duplicate elements. *[Hint: Compare $A[\lfloor n/2 \rfloor]$ and $B[\lfloor n/2 \rfloor]$. How can you reduce your search space to two sorted arrays of size $\lceil n/2 \rceil$?]*

   (b) Now suppose we are given two sorted arrays $A[1 .. m]$ and $B[1 .. n]$ with no duplicate elements and an integer k where $1 \leq k \leq m + n$. Describe an algorithm to find the kth smallest element in $A \cup B$ in $O(\log(m + n))$ time. *[Hint: Now compare $A[\lfloor m/2 \rfloor]$ and $B[\lfloor n/2 \rfloor]$ but only reduce the search space in one of the two arrays.]*

2. Suppose you are given a set $P = \{p_1, \ldots, p_n\}$ of n points in the plane, represented by two arrays $X[1 .. n]$ and $Y[1 .. n]$. Specifically, point p_i has coordinates $(X[i], Y[i])$. A point $p_i \in P$ is called ***Pareto optimal*** if there exists no point $p_j \in P$ with $i \neq j$ such that both $X[j] \geq X[i]$ and $Y[j] \geq Y[i]$. In other words, each Pareto optimal point of P has no other point of P both above and to its right.

   Describe and analyze an $O(n \log n)$ time algorithm to output the set of Pareto optimal members of P. (Any reasonable output describing these points is fine; for example, you could output an array $Z[1 .. h]$ where each element of Z is the index i of a Pareto optimal point p_i.) *[Hint: Use divide-and-conquer.]*

3. Suppose we are given an array $A[1 .. n]$ of numbers, which may be positive, negative, or zero, and which are ***not*** necessarily integers. We are going to design a dynamic programming algorithm that finds the largest sum of elements in a contiguous subarray $A[i .. j]$. For example, if we are given the array $[-6, 12, -7, 0, 14, -7, 5]$, our algorithm should return 19 for the contiguous subarray $A[2 .. 5]$. Given the one-element array $[-374]$ as input, our algorithm should return 0 (the empty interval is still an interval!) For the sake of analysis, we'll assume that comparing, adding, or multiplying any pair of numbers takes $O(1)$ time.

   (a) Unless it is empty, the maximum contiguous subarray must consist of some *last* element $A[j]$ along with 0 or more elements preceding $A[j]$. Accordingly, let $maxSum(j)$ equal the largest sum of elements in a contiguous subarray of $A[1 .. j]$ *whose last member is $A[j]$*.

   Give a recursive definition for $maxSum(j)$. Don't forget the base cases!

(b) What would be the running time of a dynamic programming algorithm that computes $maxSum(j)$ for all j from 1 to n using your recursive definition? *[Hint: You should be able to answer this question without having to describe an iterative algorithm.]*

(c) Describe and analyze an efficient algorithm that finds the largest sum of elements in a contiguous subarray of $A[1 .. n]$. *[Hint: Use parts (a) and (b).]*

(d) Now suppose in addition to $A[1 .. n]$, you are given an additional integer $X \geq 0$. Describe and analyze an algorithm that finds the largest sum of elements in a contiguous subarray of A *whose length is at most* X. *[Hint: You'll want to start over by slightly modifying the function from part (a).]*

4. For each of the following problems, the input consists of two arrays $X[1 .. k]$ and $Y[1 .. n]$ where $k \leq n$.

(a) Describe and analyze an algorithm to decide whether X is a subsequence of Y. For example, the string PPAP is a subsequence of the string PENPINEAPPLEAPPLEPEN.

(b) Suppose the input also includes a third array $C[1 .. n]$ of numbers, which may be positive, negative, or zero, where $C[i]$ is the ***cost*** of $Y[i]$. Describe and analyze an algorithm to compute the minimum cost of any occurrence of X as a subsequence of Y. That is, we want to find the minimum total cost $\sum_{j=1}^{k} C[I[j]]$ among all arrays $I[1 .. k]$ such that $I[j] < I[j+1]$ and $X[j] = Y[I[j]]$ for every index j.

(c) Describe and analyze an algorithm to compute the total number of (possibly overlapping) occurrences of X as a subsequence of Y. For purposes of analysis, assume we can add two arbitrary integers in $O(1)$ time. For example, the string PPAP appears exactly 23 times as a subsequence of the string PENPINEAPPLEAPPLEPEN. If all characters in X and Y are equal, your algorithm should return $\binom{n}{k}$.