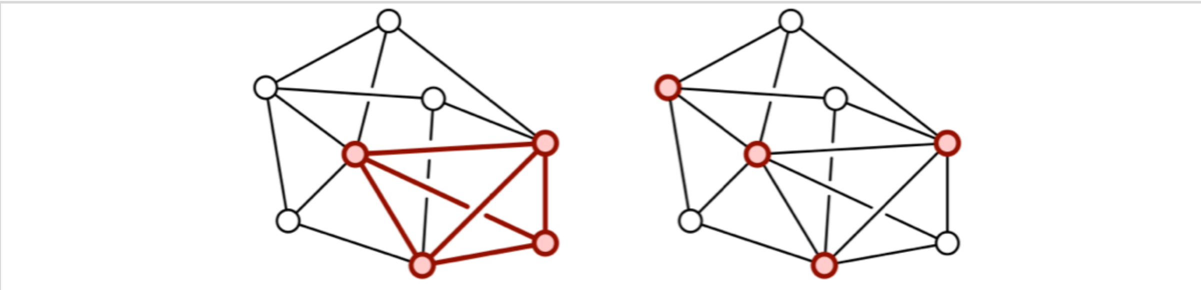
CS 6363.003.21S Lecture 23—April 29, 2021

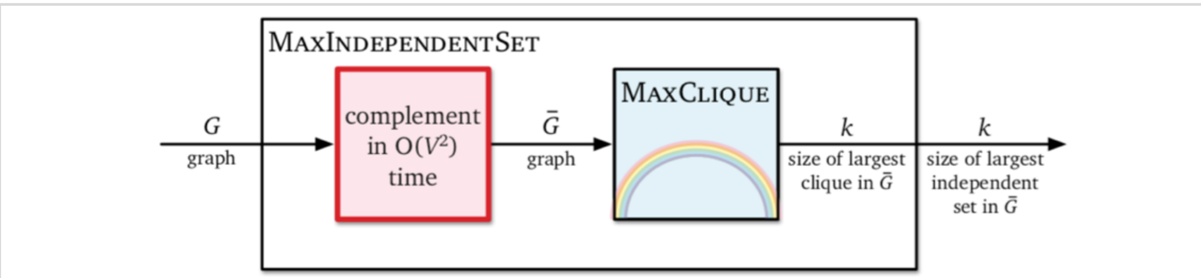
Main topics for #lecture include #NP-hardness.

Clique and Vertex Cover

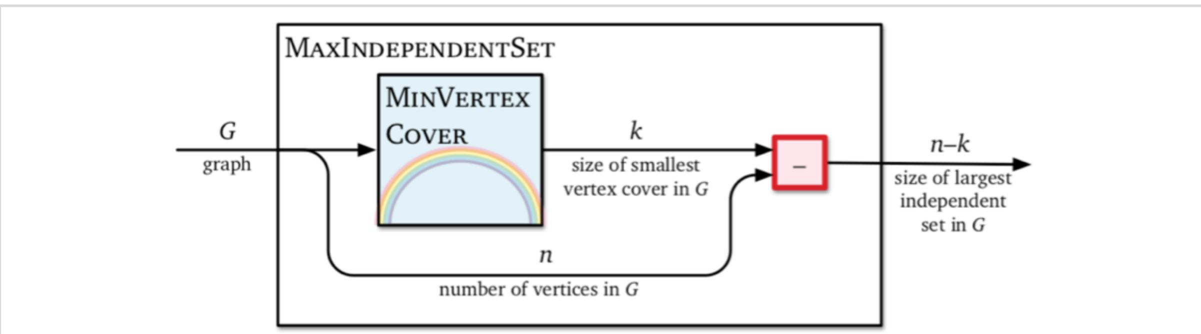
- Let's define a couple more problems. A *clique* is another name for a complete graph. The *MaxClique* problem asks for the number of vertices in the largest complete subgraph of G.
- A *vertex cover* is a set of vertices that touch every edge in the graph. *MinVertexCover* asks for the size of the smallest vertex cover in the graph.
- Below, we have a clique to the left and an vertex cover to the right.



- Claim: MaxClique and MinVertexCover are both NP-hard.
- For MaxClique, we define the edge-complement -G of G as the graph with the same vertices but the opposite set of edges so uv is an edge in -G if and only if it wasn't an edge in G.
- A set of vertices is independent in G if and only if it is a clique in -G, so we can solve MaxIndSet by solving MaxClique in the complement!



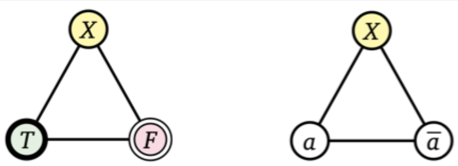
- For MinVertexCover, observe that I is an independent set in G = (V, E) if and only if V \ I is a vertex cover. So the largest independent set in G is the complement of the smallest vertex cover. If the smallest vertex cover has size k, the largest independent set has size n - k.



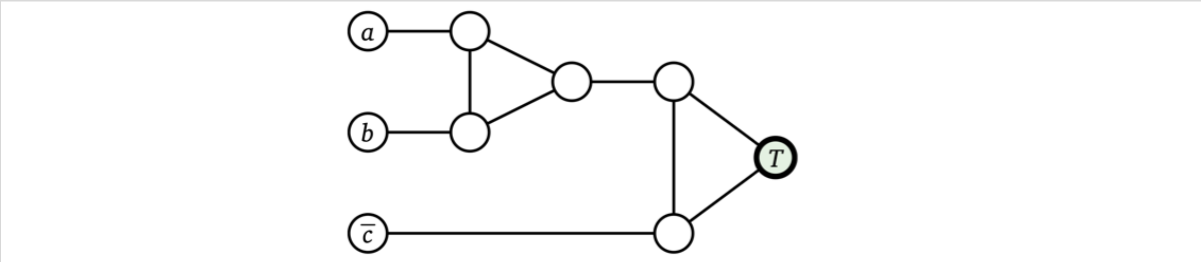
- Like before, the decisions versions of these problems are also hard. Given G and an integer k, the problems of deciding if there is a clique of size k and if there is a vertex cover of size k are both NP-complete.

Graph Coloring

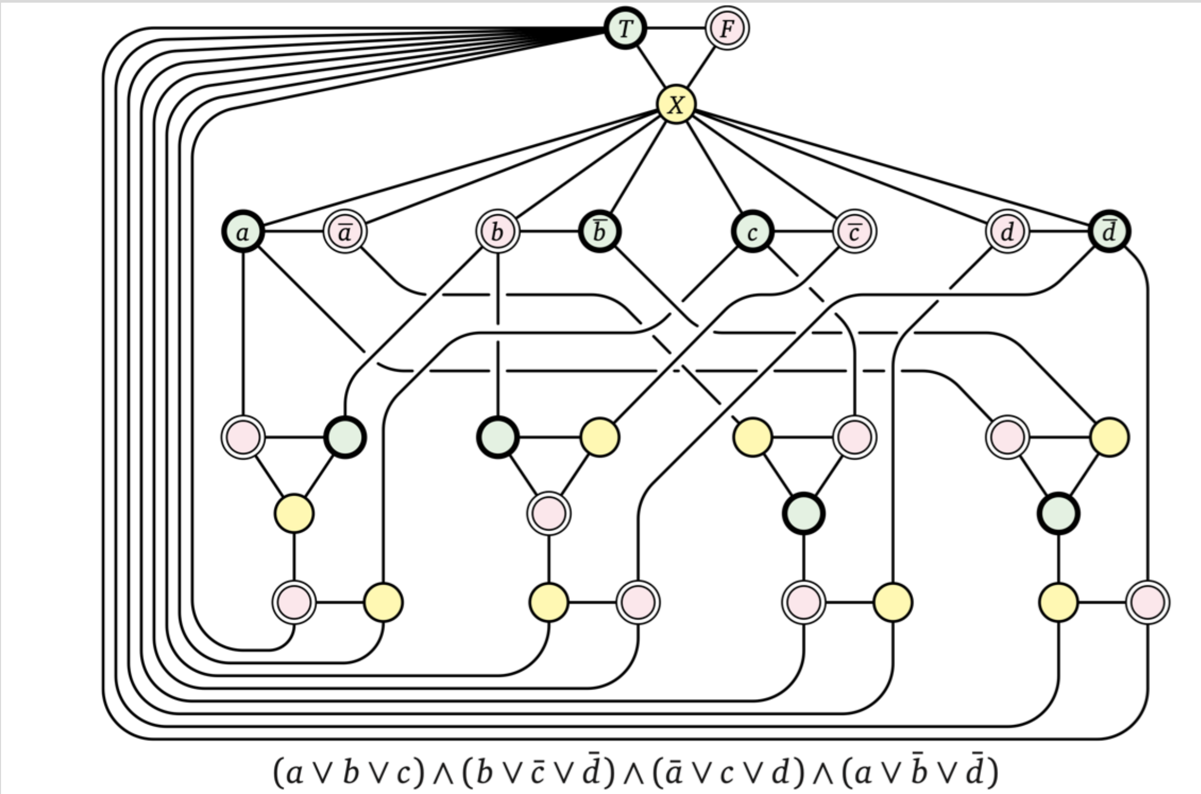
- Let's look at yet another graph problem that's a bit different.
- A *proper k-coloring* of G = (V, E) is a function C : V ➔ {1, 2, …, k} assigning one of k "colors" to each vertex so each edge has distinct colors at its endpoints.
- The graph coloring problem is to find the smallest possible number of colors to get a proper k-coloring.
- It's directly used for certain applications like compiler design. Can I store all my local variables for this function using only a few registers?
- 3Color is the "easier" problem where we simply ask, given a graph, does it have a 3-coloring?
- Claim: 3Color is NP-complete.
- It's in NP, because you can just tell me the colors and I can verify its a proper coloring in polynomial time.
- For hardness, we'll do a reduction from 3SAT. Really, this is just an example of a greater truth: If all else fails, try 3SAT.
- Suppose we're given a 3CNF formula Phi.
- In many reductions, we build an input to the new problem by combining together a collection of useful subgraphs called *gadgets*. There are three types of gadget for 3Color:
    - A *truth gadget*: A triangle with vertices T, F, and X standing for True, False, and Other. These three vertices have to have different colors in a proper 3-coloring. For convenience, we'll refer to the colors they get as True, False, and Other respectively.
    - For each variable a, a *variable gadget*: a triangle joining two vertices a and ~a to the same vertex X used in the truth gadget. Vertex a must be colored True or False in proper 3-coloring, implying ~a gets False or True, respectively.



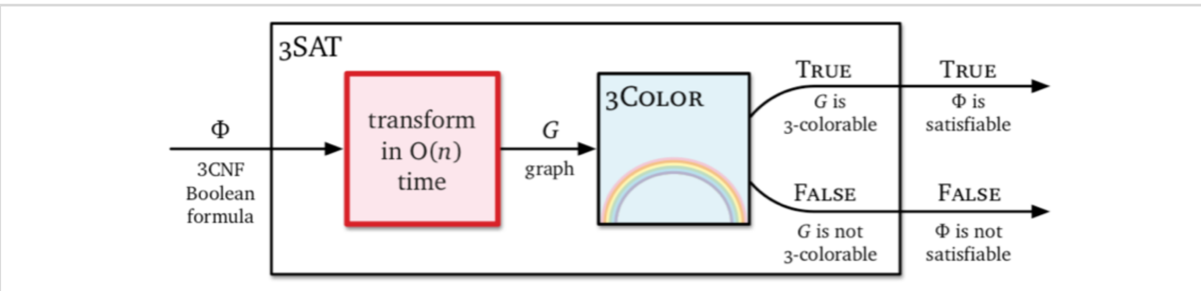
    - For each clause in Phi, a *clause gadget*: We join the three literal vertices for the clause to vertex T using five new vertex and ten new edges. Tedious case analysis implies that if a 3-coloring gives all three literals the color False, then there is a monochromatic edge in the clause gadget. However, it is always possible to color at least one literal True and the rest False and then extend those colors to a proper 3-coloring of the whole clause gadget.



- Here's a whole example. Clearly, we can build it in polynomial time.



- I claim the graph is 3-colorable if and only if Phi is satisfiable.
  - If the graph is 3-colorable, exactly one of a or ~a is assigned True for each variable. And as I already argued, at least one literal per clause is assigned True. So we can give the literals with color True an assignment of True to satisfy Phi.
  - If Phi is satisfiable, then we color each True literal with the color True and each False literal with the color False. We extend these choices to a proper 3-coloring of each clause gadget to get a proper 3-coloring of the whole graph.
- Here's the whole algorithm:



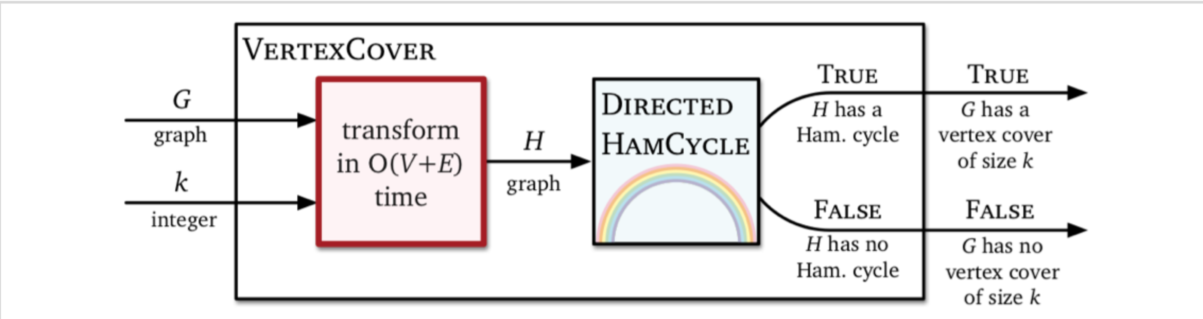
- 3Color is NP-hard since we got a reduction, so it's NP-complete.
- And as you might expect, for any constant k > 3, the analogously defined problem kColor

is NP-complete. We can even reduce from 3Color.

- And the more general optimization version of graph coloring "how many colors do I need" must be NP-hard as well, since it naturally solves 3Color.

Hamiltonian Cycle

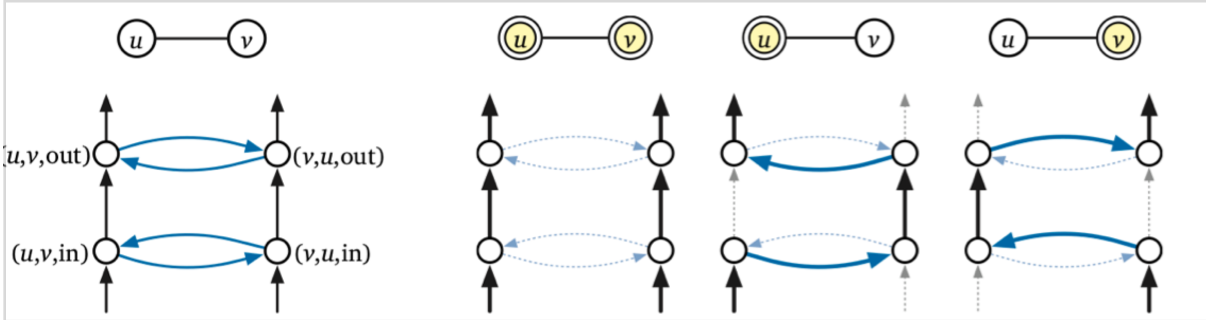
- A Hamiltonian cycle is a cycle in a graph that visits every *vertex* exactly once. (Visiting every edge once is an Eulerian tour.)
- The Hamiltonian cycle problem is given a directed graph H = (V, E), does H contain a Hamiltonian cycle?
- Claim: Hamiltonian cycle is NP-complete.
- (Incidentally, the Eulerian tour problem is in P.)
- I can tell you the order of vertices in the cycle so it must be in NP.
- We'll reduce from the decision version of VertexCover: is there a vertex cover of size k?
- I'll be a bit lighter on the details here so we can get to one last example.
- Let's start with the box algorithm this time. We want to know if undirected graph G has a vertex cover of size k, so we'll build a directed graph H and ask if it has a Hamiltonian cycle.



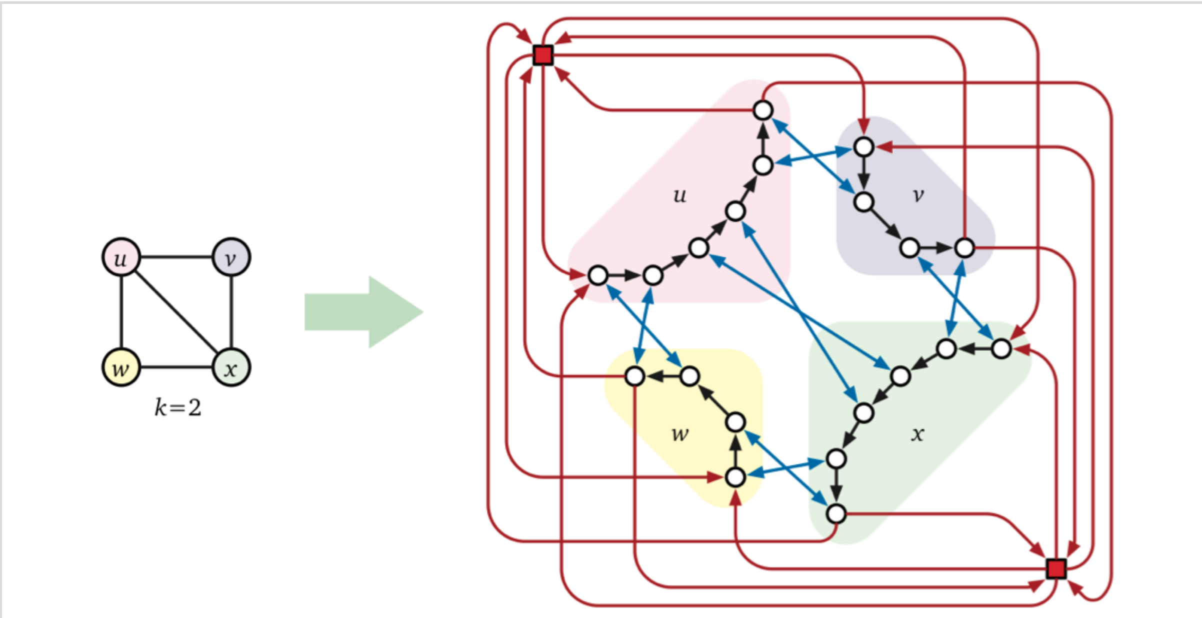
- Now let's build some gadgets.
- Each edge uv in G becomes four vertices (u, v, in), (u, v, out), (v, u, in), and (v, u, out) in H plus six directed edges

$$
\begin{array}{ccc}
(u,v,\mathrm{in})\to(u,v,\mathrm{out}) & (u,v,\mathrm{in})\to(v,u,\mathrm{in}) & (v,u,\mathrm{in})\to(u,v,\mathrm{in}) \\
(v,u,\mathrm{in})\to(v,u,\mathrm{out}) & (u,v,\mathrm{out})\to(v,u,\mathrm{out}) & (v,u,\mathrm{out})\to(u,v,\mathrm{out})
\end{array}
$$

- Eventually, we'll have one edge going out from each out vertex and one edge going in to each in vertex as shown below. There's exactly three routes a Hamiltonian cycle can take through the four vertices. Each will correspond to a choice of u and v, only u, or only v being used in a vertex cover of G.



- For each vertex u in G, we connect all the edge gadgets for edges uv in a directed path called the *vertex chain*. Specifically, if u has d neighbors v_1, v_2, …, v_d, we add d - 1 edges (u, v_i, out) ➡ (u, v_{i+1}, in) for each i from 1 to d - 1.
- And we add k additional *cover vertices* x_1, x_2, …, x_k. Each has an edge to the first in vertex in each chain and an edge from the last out vertex in each chain. Here's an example:



- Now, suppose there is a vertex cover u_1, u_2, …, u_k in G. We can find a Hamiltonian cycle in H. For each i from 1 to k, we go from x_i, through u_i's chain, and to x_{i+1}. We go through each of u_i's edge gadgets as described above, detouring through the (v, u_i) vertices for any edge uv where v is not in the cover.
- Conversely, suppose there is a Hamiltonian cycle. It must contain an edge from every cover vertex to the start of some vertex chain. If you start at some edge (u, v, in) you must leave (u, v, out), so we'll touch every edge gadget in the chain. Every edge gadget for uv must be entered in at least one of its two entrances (u, v, in) or (v, u, in), so we'll go through enough vertex chains to cover all edges in G. And we go through exactly k vertex chains, so we have a vertex cover of size k.
- There are many variants of HamiltonianCycle. HamiltonianPath asks if there is a path containing each vertex exactly once. It's also NP-hard. These problems remain NP-hard in undirected graphs as well.
- And if you're interested, Erickson and CLRS also show reductions from 3Sat directly to Hamiltonian cycle.

Subset Sum

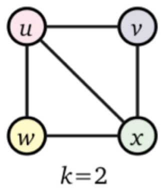
- Let's finish with one last problem that doesn't involve booleans or graphs.
- SubsetSum: Given a set X of positive integers and an integer T, does X have a subset whose elements sum to T?
- We'll again reduce from VertexCover. Suppose we have a graph G = (V, E) and an integer k. Is there a vertex cover of size k?
- We need to make a set X of integers and a target value T so there's a subset sum of T if and only if there's a vertex cover of size k.
- We're still going to use gadgets, but now the gadgets will be very large numbers.
- Number the edges of G from 0 to |E| - 1. X contains b_i := 4^i for each edge i.
- For each vertex v, it also contains

$$a_v := 4^E + \sum_{i \in \Delta(v)} 4^i$$

  where Delta(v) is the set of edges incident to v.
- We can also imagine each integer as written in base 4 using (E + 1)-digits. The |E|th digit (quit?) is 1 if the integer represents a vertex and is 0 otherwise.
- For each i < |E|, the ith digit is 1 if the integer represents edge i or one of its endpoints, and is 0 otherwise.
- Finally, set

$$T := k \cdot 4^E + \sum_{i=0}^{E-1} 2 \cdot 4^i.$$

- It only takes polynomial time to write out these numbers in base 4 or even binary.
- Here's the SubsetSum instance for that graph we were using for HamiltonianCycle.



$$\begin{aligned}
a_u &:= 111000_4 = 1344 & b_{uv} &:= 010000_4 = 256 \\
a_v &:= 110110_4 = 1300 & b_{uw} &:= 001000_4 = 64 \\
a_w &:= 101101_4 = 1105 & b_{vw} &:= 000100_4 = 16 \\
a_x &:= 100011_4 = 1029 & b_{vx} &:= 000010_4 = 4 \\
& & b_{wx} &:= 000001_4 = 1
\end{aligned}$$

  T := 222222_4 = 2730
- And here's a good example of why we need to do the full proof correctness, because it's still not clear how a subset summing to T corresponds to a vertex cover!
- Suppose there is a vertex cover C of size k.

- Let X_C be the subset of integers that contains a_v for every vertex v in C and b_i for every edge i covered *exactly once*.
  - The sum of the integers, written in base 4, has a 2 in each of the first |E| digits.
  - And we're summing k 4^|E|'s for our k vertices.
  - So the sum is exactly T.
- Suppose there is a subset X' of X that sums to T.
  - Let V' be the subset of vertices whose vertex numbers we chose.
  - There are no carries in the first E digits, because for each i there are only three numbers whose ith digit is 1. Each edge number b_i contributes a single 1 to the ith digit, so we need one of the vertex numbers for its endpoints. In other words, each edge is covered by V'.
  - And every vertex number is at least 4^|E|, so we can only afford to use k of them. |V'| ≤ k.
- Now, if you were reading carefully, you might have noticed there's an O(nT) time algorithm for SubsetSum earlier in Erickson. Doesn't that imply P = NP?
- That's another example of a *pseudo-polynomial time* algorithm, like Ford-Fulkerson with arbitrary integer capacities. The worst-case running time is exponential in the actual input size, because we only used O(log T) bits to write out these numbers.
- NP-hard problems with such algorithms are called *weakly NP-hard*. If the problem is still NP-hard even when writing down any numbers in unary (so in space proportional to their value), then the problem is called *strongly NP-hard*. SubsetSum is the only example we've seen of a weakly NP-hard problem. Another famous one is the Partition problem, where you try to partition a set of integers into two pieces with equal sums.
- And that's it! I'll send a reminder, but I think instead of meeting Tuesday, I'll give you a chance to take a practice QE exam. We can discuss how it went on Thursday. And I'll do another regular review session sometime before your final on Thursday May 13th.
- And in case I forget to ask later, please fill out evaluations for the course at eval.utdallas.edu. I do take your comments seriously when thinking about how I can improve this and other courses. Comments and scores you give affect things like raises and tenure, so whether or not you liked the course or my instruction, I think it's in yours and others' interest to give honest evaluations.