Given directed graph
\[ G = (V, E) \] with weight function
\[ w : E \to \mathbb{R}. \]
The shortest path from \( s \) to \( t \) is the \textit{st}-path \( P \) minimizing
\[ \sum_{u,v \in P} w(u \to v). \]
\[ \uparrow \text{no repeated vertices} \]
most distance algs solve
\[ \text{single source shortest paths (SSSP) problem} \] find all shortest paths from \( s \)
every subpath of a shortest path is shortest

pick paths consistently so they lie on a tree rooted at s

we'll find this tree
in an undirected graph

\[ a \quad w(e) \quad \sqrt{\text{V}} \quad \Rightarrow \]

turn it directed by making two orientations of each edge.

MST  SSSP
Edge weights may be negative! All efficient algorithms really look for shortest walks.

There is no shortest walk if you can take a negative length cycle.

Is no neg. cycle $\Rightarrow$ shortest walks are paths $\Rightarrow$ you will find shortest path
The One Algorithm [Ford, Dantzig, Minty]:

For each $v \in V$, keep two mutable variables

$\text{dist}(v)$: our guess on distance from $s$ to $v$. Always $\geq$ the actual distance.

Initially, $\text{dist}(s) \leftarrow 0$

$\text{dist}(v) \leftarrow \infty$ \text{forall}$v \neq s$

$\text{pred}(v)$: predecessor vertex on some tentative shortest $s$-$t$-walk.

The "proof" that $\text{dist}$ was too high. Initially: $\text{pred}(v) \leftarrow \text{Null}$.
All SSSP algs begin with...

**INIT-SSSP(s):**

- \( \text{dist}(s) \leftarrow 0 \)
- \( \text{pred}(s) \leftarrow \text{NULL} \)
- for all vertices \( v \neq s \)
  - \( \text{dist}(v) \leftarrow \infty \)
  - \( \text{pred}(v) \leftarrow \text{NULL} \)

Can edge \( u \rightarrow v \) tense if

\[
\text{dist}(u) + w(u \rightarrow v) < \text{dist}(v)
\]

\[\uparrow\]

**proof that dist\( (v) \) is too high**

Relax it just enough to remove the tension.

**RELAX\( (u \rightarrow v) \):**

- \( \text{dist}(v) \leftarrow \text{dist}(u) + w(u \rightarrow v) \)
- \( \text{pred}(v) \leftarrow u \)
FordSSSP(s):
InitSSSP(s)
while there is at least one tense edge
RELAX any tense edge

If no neg. cycles...
FordSSSP terminates with
each \( \text{dist}(v) = \) distance to \( v \)
each \( \text{pred}(v) = \) predecessor on shortest path
\( \text{dist}(v) = \infty \) if \( v \) unreachible

If neg. cycles reachable from \( s \)
FordSSSP never terminates!
Lemma: At all times, for any vertex \( v \), \( \text{dist}(v) = \infty \) or it equals the length of a walk from \( s \) to \( t \).

Proof: Induction on \( \# \) relaxations.

If \( \text{dist}(v) \) never changed, either \( \text{dist}(v) = \infty \) or \( v = s \) so \( \text{dist}(v) \) is length of trivial \( ss \)-walk.

O.w.

At previous change, we set \( \text{dist}(v) \leq \text{dist}(u) + w(u,v) \) for some \( u \), \( \text{dist}(u) = \) length of some \( s, u \)-walk.
Add $u \Rightarrow v$ to walk for an $s,v$-walk of length $\text{dist}(v)$
Bellman-Ford: If all else fails.

BellmanFord(s)
InitSSSP(s)
while there is at least one tense edge
for every edge $u \rightarrow v$
if $u \rightarrow v$ is tense
RELAX($u \rightarrow v$)

$\text{dist}_i(v)$: length of shortest walk to $v$ that has at most $i$ edges.

So... $\text{dist}_0(s) = 0$
$\text{dist}_0(v) = \infty \forall v \neq s$
Lemma: For every vertex $v$ and every non-negative integer $i$, after $i$ iterations of the while loop $\text{dist}(v) \leq \text{dist}(v)$.

Proof: If $i = 0$, \[ \checkmark \]

O.w., let $W$ be shortest $sv$-walk with $\leq i$ edges. If $W$ has no edges, it is the trivial $ss$-walk, so $v = S \text{ and } \text{dist}(v) = 0$. $\text{dist}(s) < 0$ initially and $\text{dist}$ values only decrease. \[ \checkmark \]
o.w. let \( u \rightarrow v \) be last edge of \( W \)

After \( i - 1 \) iterations,
\[
\text{dist}(u) = \text{dist}^{-1}(u) \leq i - 1
\]

In \( i \)th iteration, we considered \( u \rightarrow v \),

- either \( \text{dist}(v) = \text{dist}(u) + W(u \rightarrow v) \)
- or - we set \( \text{dist}(v) \leftarrow \text{dist}(u) + W(u \rightarrow v) \)
either way...

\[
dist(u) 
\leq dist(v) \leq dist(u) + w(u \to v) 
\leq dist(u) + w(u \to v) 
\leq dist(v) 
\]

\[dist(v)\] can only decrease after that

Lemma is true even with neg. cycles!
\( \text{dist}(u) \) never < distance to \( v \).

Distance to \( v = \text{dist}(u) \) if no neg. cycles

All paths have \( \leq V-1 \) edge

so \( \text{dist}(v) = \text{distance to } v \) after \( \leq V-1 \) iterations

Each iteration of while loop takes \( O(E) \) time.

\( O(VE) \) time!
**BellmanFord(s)**

**InitSSSP(s)**

repeat $V - 1$ times

for every edge $u \rightarrow v$

if $u \rightarrow v$ is tense

RELAX($u \rightarrow v$)

for every edge $u \rightarrow v$

if $u \rightarrow v$ is tense

return “Negative cycle!”

Still $O(VE)$ time.