Given directed graph $G=(V,E)$ + a weight function $w: E \to \mathbb{R}$.

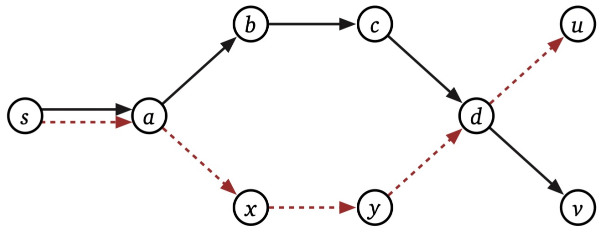
The shortest path from s to t is the s,t-path P minimizing $\sum_{u \to v \in P} w(u \to v)$.

no repeated vertices

distance from s to t

most distance algs solve single source shortest paths (SSSP) problem: find all shortest paths from s

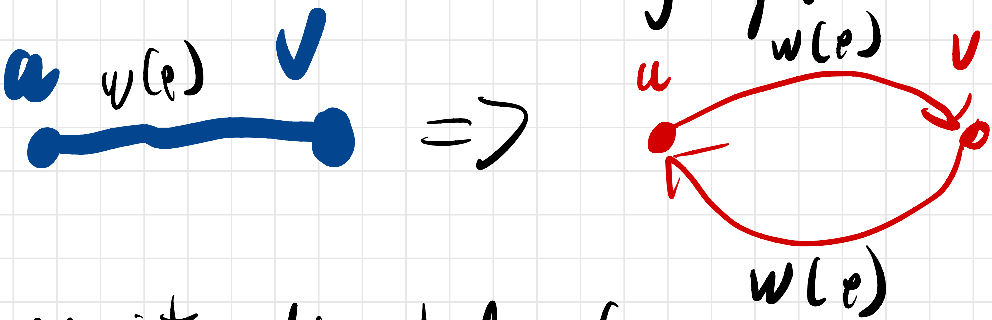
every subpath of a shortest path is shortest



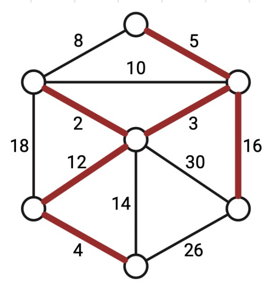
pick paths consistently so they lie on a tree rooted at s

we'll find this tree

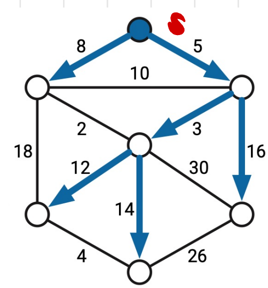
in an undirected graph



turn it directed by making two orientations of each edge.



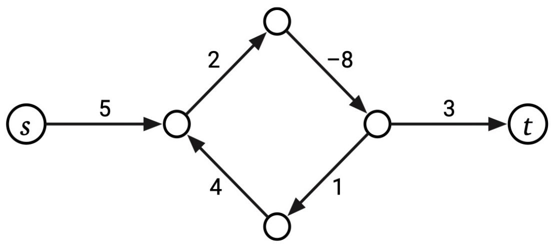
MST



SSSP

Edge weights may be negative!

All efficient algorithms really look for shortest <u>walks</u>.



There is no shortest walk if you can take a negative length cycle.

If no neg. cycle ⟹ shortest walks are paths ⟹ you will find shortest path

The One Algorithm [Ford, Dantzig, Minty]:

For each $v \in V$, keep two mutable variables

$dist(v)$: our guess on distance from s to v. Always $\geq$ the actual distance.

Initially, $dist(s) \leftarrow 0$

$dist(v) \leftarrow \infty \ \forall v \neq s$

$pred(v)$: predecessor vertex on some tentative shortest s,t-walk. The "proof" that dist was too high. Initially: $pred(v) \leftarrow$ Null $\forall v$

All SSSP algs begin with...

INITSSSP(s):
  $dist(s) \leftarrow 0$
  $pred(s) \leftarrow$ NULL
  for all vertices $v \neq s$
    $dist(v) \leftarrow \infty$
    $pred(v) \leftarrow$ NULL

Call edge $u \to v$ tense if

$$dist(u) + w(u \to v) < dist(v)$$

proof that dist(v) is too high

Relax it just enough to remove the tension.

RELAX($u \to v$):
  $dist(v) \leftarrow dist(u) + w(u \to v)$
  $pred(v) \leftarrow u$

```
FordSSSP(s):
    InitSSSP(s)
    while there is at least one tense edge
        Relax any tense edge
```

If no neg. cycles...

FordSSSP terminates with

each $dist(v)$ = distance to v

each $pred(v)$ = predecessor on shortest path

($dist(v) = \infty$ if v unreachible)

If neg. cycles reachible from s

FordSSSP never terminates!

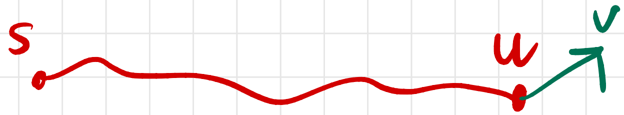
Lemma: At all times, for any vertex v, $dist(v) = \infty$ or it equals the length of a walk from s to t.

Proof: Induction on # relaxations.

If $dist(v)$ never changed, either $dist(v) = \infty$ or $v = s$ & $dist(v) =$ length of trivial s,s-walk.

o.w.

at previous change, we set $dist(v) \leftarrow dist(u) + w(u \to v)$ for some u. $dist(u) =$ length of some s,u-walk.



Add $u \to v$ to walk for an s,v-walk of length $\mathrm{dist}(v)$ ✓

Bellman-Ford: If all else fails.

```
BELLMANFORD(s)
  INITSSSP(s)
  while there is at least one tense edge
      for every edge u→v
          if u→v is tense
              RELAX(u→v)
```

$dist_{\leq i}(v)$: length of shortest walk to v that has at most i edges.

So... $dist_{\leq 0}(s) = 0$

$dist_{\leq 0}(v) = \infty \quad \forall v \neq s$

Lemma: For every vertex v & every non-neg integer i, after i iterations of while loop $dist(v) \leq dist_{\leq i}(v)$.

Proof: If $i=0$, ✓

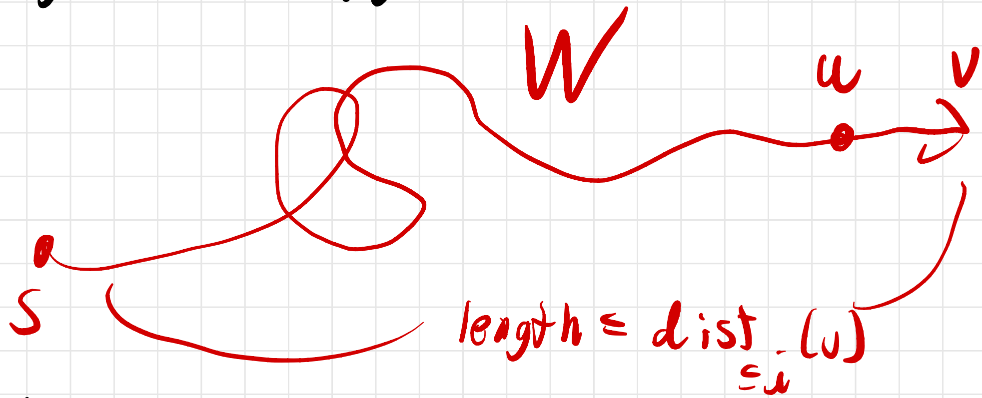
o.w.,

Let W be shortest s,v-walk with $\leq i$ edges.

If W has no edges, it is the trivial s,s-walk, so $v=s$ & $dist_{\leq i}(v) = 0$.

$dist(s) \leftarrow 0$ initially & dist values only decrease. ✓

o.w. let $u \to v$ be last edge of W



After $i-1$ iterations,

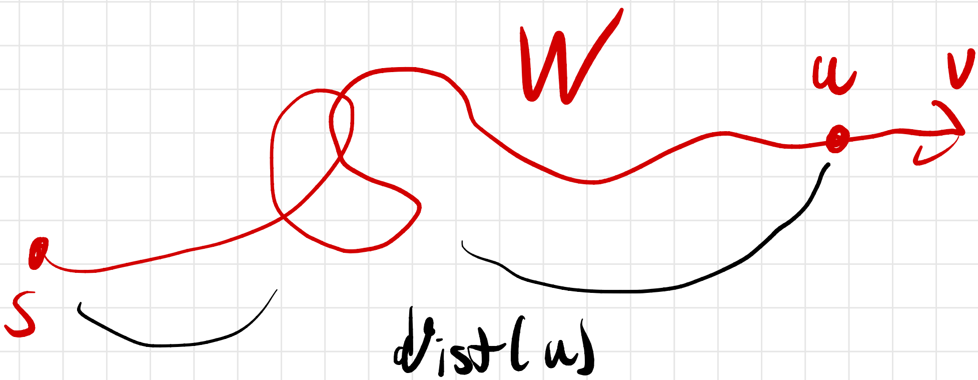
$$dist(u) \le dist_{\le i-1}(u)$$

In ith iteration, we considered $u \to v$.

either $dist(v) \le dist(u) + w(u \to v)$

-or- we set $dist(v) \leftarrow dist(u) + w(u \to v)$

either way...



$$\mathrm{dist}(v) \le \overset{\le i-1}{\mathrm{dist}(u)} + w(u \to v)$$

$$\le \mathrm{dist}_{\le i-1}(u) + w(u \to v)$$

$$= \mathrm{dist}_{\le i}(v)$$

dist(v) can only decrease after that

Lemma is true even with neg. cycles!

dist(v) never < distance to v.

distance to v $\le$ dist(v)
if no neg. cycles $\le |V|-1$

all paths have $\le V-1$ edge

so dist(v) = distance to v
after $\le V-1$ iterations

Each iteration of while loop takes O(E) time.

O(VE) time!

```
BELLMANFORD(s)
    INITSSSP(s)
    repeat V − 1 times
        for every edge u→v
            if u→v is tense
                RELAX(u→v)
    for every edge u→v
        if u→v is tense
            return "Negative cycle!"
```

Still $O(VE)$ time.