SECRET RM-1573 10-24-55 -33-

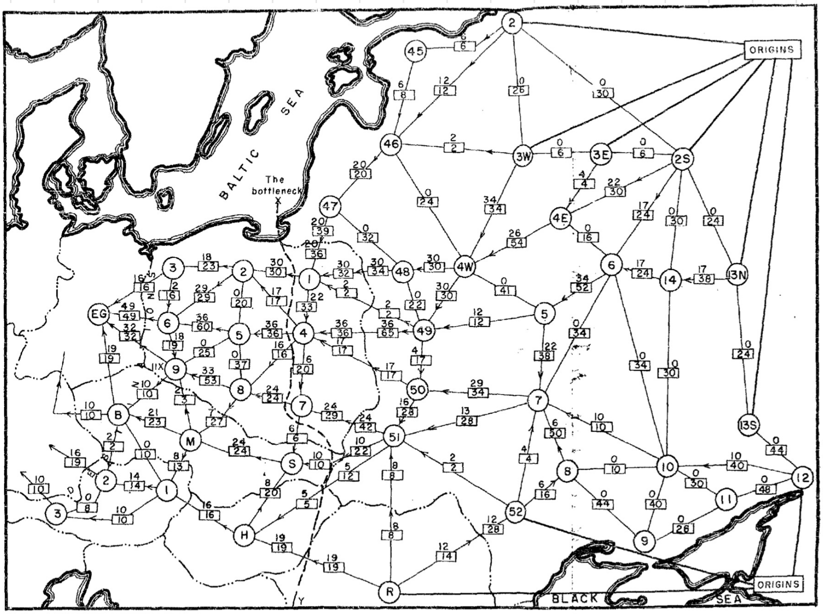


Fig. 7 — Traffic pattern: entire network available

Legend:

International boundary

(B) Railway operating division

9/12 Capacity: 12 each way per day. Required flow of 9 per day toward destinations (in direction of arrow) with equivalent number of returning trains in opposite direction

All capacities in trains/1000's of tons each way per day

Origins: Divisions 2, 3W, 3E, 2S, 13N, 13S, 12, 52 (USSR), and Roumania

Destinations: Divisions 3, 6, 9 (Poland); B (Czechoslovakia); and 2, 3 (Austria)

Alternative destinations: Germany or East Germany

Note IIX at Division 9, Poland

Two problems:

Given directed graph $G = (V, E)$ + two vertices

$s \neq t$

Source

Target/sink

Maximum Flow

(s,t)-flow $f: E \to \mathbb{R}_{\geq 0}$ that satisfies <u>conservation constraints</u>:

$$\forall v \in V, \ v \neq s,t$$

$$\sum_{u} f(u \to v) = \sum_{w} f(v \to w)$$

total in = total out

If $u \to v \notin E$, say $f(u \to v) = 0$

$$\partial f(v) := \sum_{w} f(v \to w) - \sum_{u} f(u \to v)$$

$$\Rightarrow \partial f(v) = 0 \quad \forall v \neq s,t$$

The <u>value</u> of f is

$$|f| := \partial f(s) = \sum_w f(s \to w) - \sum_u f(u \to s)$$

$$\sum_v \partial f(v) = \partial f(s) + \partial f(t)$$

$$\Rightarrow 0 = \partial f(s) + \partial f(t)$$

$$\Rightarrow \partial f(t) = -|f|$$

Also give a <u>capacity</u> function

$$c : E \to \mathbb{R}_{\geq 0}$$

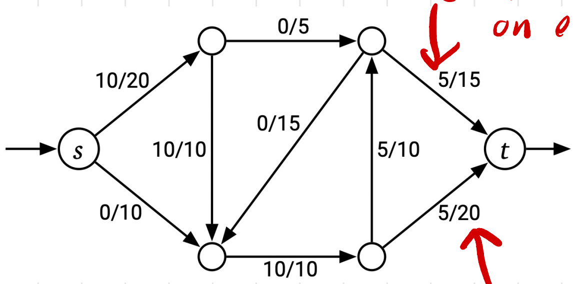
f is <u>feasible</u> wrt c if

$$f(e) \leq c(e) \ \forall e$$

f **saturates** e if $f(e) = c(e)$

f **avoids** e if $f(e) = 0$

flow on edge



capacity

maximum flow problem:

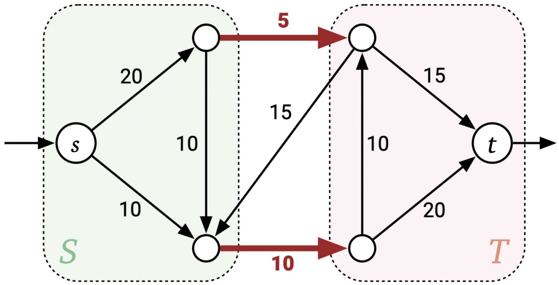
find a max value (s,t)-flow that is feasible wrt c.

An **(s,t)-cut** is a **partition** of V into two subsets ($S \cup T = V$ + $S \cap T = \emptyset$) s.t. $s \in S$ + $t \in T$.

Given $c: E \to \mathbb{R}_{\geq 0}$, the **capacity** of cut (S, T) is sum of capacities for edges from S to T

$$\|S, T\| := \sum_{v \in S} \sum_{w \in T} c(v \to w)$$

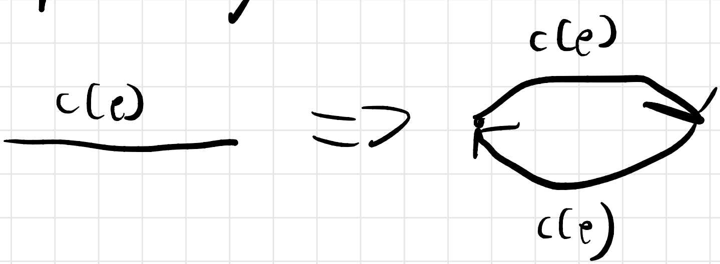
(if $v \to w \notin E$, say $c(v \to w) = 0$)



$$\|S,T\| = 15$$

minimum cut problem:

find (s,t)-cut of minimum capacity



Lemma: The value of any (s,t)-flow f is at most the capacity of any (s,t)-cut (S,T).

$$
\begin{aligned}
|f| &= \partial f(s) && \text{[by definition]}\\
&= \sum_{v\in S} \partial f(v) && \text{[conservation constraint]}\\
&= \sum_{v\in S}\sum_{w} f(v\to w) - \sum_{v\in S}\sum_{u} f(u\to v) && \text{[math, definition of } \partial\text{]}\\
&= \sum_{v\in S}\sum_{w\notin S} f(v\to w) - \sum_{v\in S}\sum_{u\notin S} f(u\to v) && \text{[removing edges from } S \text{ to } S\text{]}\\
&= \sum_{v\in S}\sum_{w\in T} f(v\to w) - \sum_{v\in S}\sum_{u\in T} f(u\to v) && \text{[definition of cut]}\\
&\le \sum_{v\in S}\sum_{w\in T} f(v\to w) && \text{[because } f(u\to v)\ge 0\text{]}\\
&\le \sum_{v\in S}\sum_{w\in T} c(v\to w) && \text{[because } f(v\to w)\le c(v\to w)\text{]}\\
&= \|S,T\| && \text{[by definition]}
\end{aligned}
$$

So... $|f| = \|S,T\|$ if and only if f saturates all S→T edges + avoids all T→S edges.

$\Rightarrow$ f is maximum value &

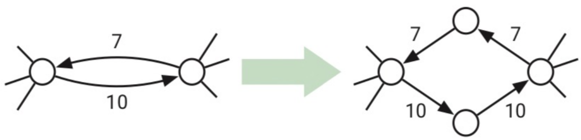
(S, T) is minimum capacity

Maxflow Mincut Theorem

[Ford-Fulkerson '54, Elias, Feinstein, Shannon '56]:

The value of the max flow = capacity min cut!

Assume the graph is <u>reduced</u>: one of $u \to v$ or $v \to u$ not in E



Suppose we have a feasible (s,t)-flow f.

Residual capacities wrt f

$$c_f : V \times V \to \mathbb{R}$$

$$c_f(u \to v) = \begin{cases} c(u \to v) - f(u \to v) & \text{if } u \to v \in E \\ f(v \to u) & \text{if } v \to u \in E \\ 0 & \text{o.w.} \end{cases}$$

how much we can undo

$f(u \to v) \geq 0$

$f(u \to v) \leq c(u \to v)$

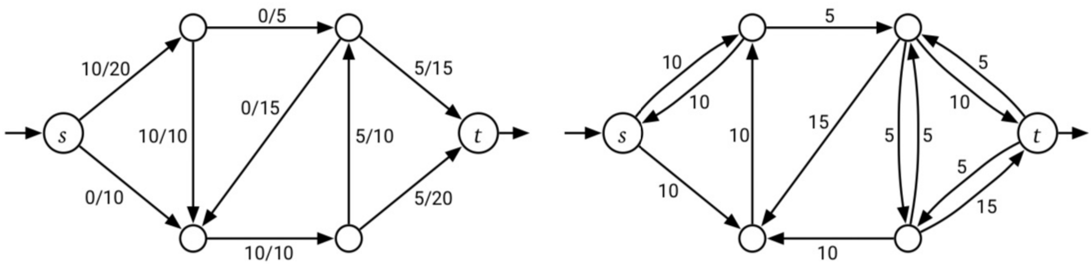
so c_f is non-negative

$f(u \to v)$ may be > 0 even if $u \to v \notin E$

<u>residual graph</u> $G_f = (V, E_f)$

E_f : pairs $u \to v$ s.t. $c_f(u \to v) > 0$

positive



Suppose $\exists$ path P from s to t in G_f.

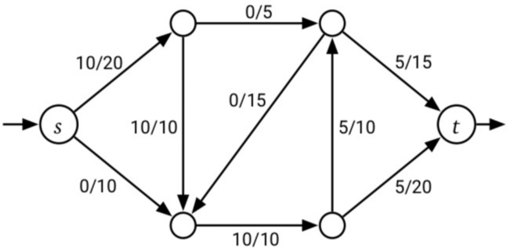
augmenting path

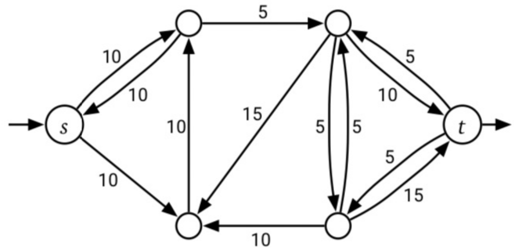
max amount of flow we can "push" through P

$$F := \sum_{u \to v \in P} c_f(u \to v)$$

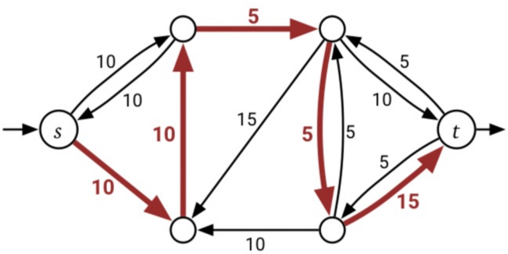
Define $f' : E \to \mathbb{R}$

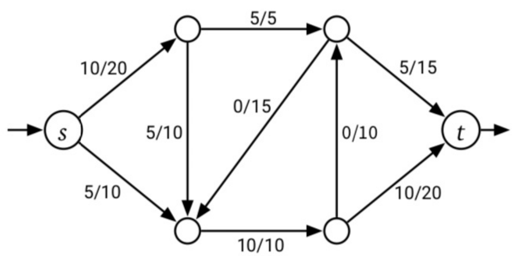
$$f'(u \to v) = \begin{cases} f(u \to v) + F & \text{if } u \to v \in P \\ f(u \to v) - F & \text{if } v \to u \in P \\ f(u \to v) & \text{o.w} \end{cases}$$





F = 5





f' is a feasible (s,t)-flow

$|f'| = |f| + F \Rightarrow f$ was not max value

Suppose o.w. that s cannot reach t in G_f.

S: vertices reachable from s in G_f

T: $V \setminus S$.

(S, T) is an (s,t)-cut.

$\forall u \in S, v \in T$

saturated

If $u \to v \in E$, then

$$0 = c_f(u \to v) = c(u \to v) - f(u \to v)$$

avoided

If $v \to u \in E$, then

$$0 = c_f(u \to v) = f(v \to u)$$

$$\Rightarrow |f| = \|S, T\|$$

max

min