max flow = min cut

Given $G=(V,E)$, two vertices $s \neq t$, capacities $c: E \to \mathbb{R}_{\geq 0}$

proof

f: arbitrary feasible (s,t)-flow

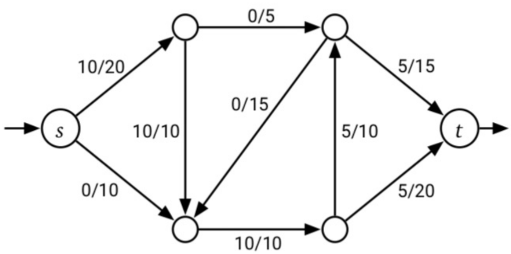
Built residual graph $G_f=(V,E_f)$

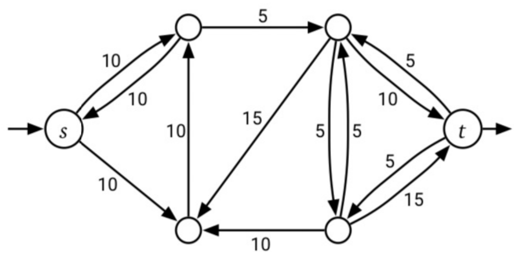
$E_f \subseteq V \times V$: pairs $u \to v$ with positive residual capacity

$c_f: V \times V \to \mathbb{R}_{\geq 0}$

$$c_f(u \to v) = \begin{cases} c(u \to v) - f(u \to v) & \text{if } u \to v \in E \\ f(v \to u) & \text{if } v \to u \in E \\ 0 & \text{otherwise} \end{cases}$$

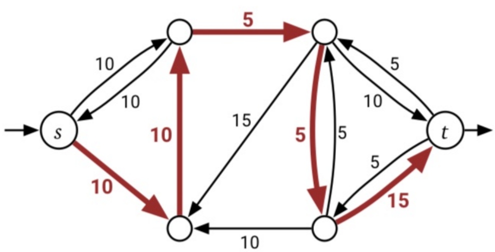
$\left(|f| = \sum_{s \to w} f(s \to w) - \sum_{u \to s} f(u \to s)\right)$

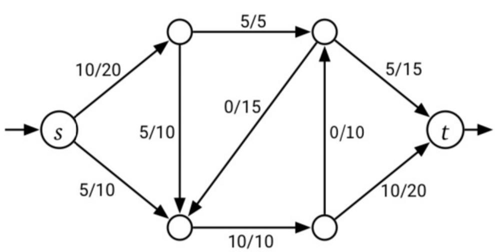




G $\qquad\qquad\qquad$ G_f

If $\exists$ an augmenting path P from s to t in G_f, push on P to make flow f' where $|f'| > |f|$.



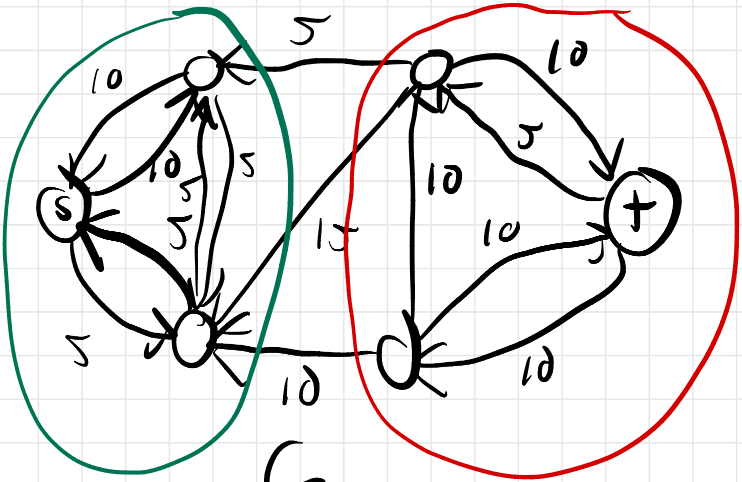


$(|f'| = |f| + 5)$

If no (s,t)-path in G_f,

$S :=$ all vertices reachible from s in G_f. $T := V \setminus S$.

max flow $\rightarrow |f| = \|S,T\| \leftarrow$ min cut so done



$G_{f'}$

S ← reachable from s

T

Ford-Fulkerson Augmenting Path Algorithm

- start with $f(u \to v) := 0$ $\forall u \to v \in E$
- repeatedly try to push along augmenting paths in residual graph

Assume integer capacities...

- flow is intially intregal
- integral flow $\Rightarrow$ integral c_f
- $\Rightarrow$ push integer amount of flow

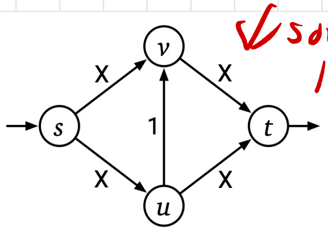
$\Rightarrow$ new flow f' is integral & $|f'| \geq |f| + 1$.

f^*: some maximum flow

we do at most $|f^*|$ pushes & end with an integral flow

$O(E)$ time to build G_f & push

so $O(E|f^*|)$ (integer capacities)



some large int

$|f^*| = 2X$. So $O(X)$ time.

Input needs only $O(\log X)$ bits.

exponential time! "?"

(pseudo-polynomial time!)

May not terminate with real ~~o~~ capacities.

But, may be fast in practice or if $|f^*|$ is small.

Edmonds-Karp choices for augmenting paths:

1) Fattest Augmenting Paths

(largest bottleneck capacity)

$O(E \log V)$ time to find path

f: current flow

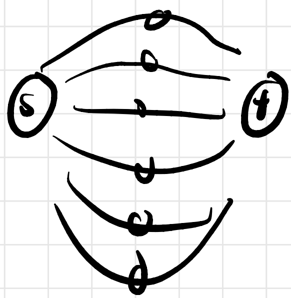
f': max flow in G_f

($f + f'$ is f^*)

e: bottleneck edge. we push $c_f(e)$ units of flow

We can decompose f' into $\leq |E|$ path flow + cycles,

$c_f(e) \geq |f'| / |E|$



$\Rightarrow$ so residual max flow value is now $\leq \left(1 - \frac{1}{|E|}\right)|f|$

After $|E| \cdot \ln|f^*|$ iterations...

we still need to push $\leq$

$$|f^*| \cdot \left(1 - 1/|E|\right)^{|E| \cdot \ln|f^*|} \leq |f^*| \cdot e^{-\ln|f^*|} = 1$$

If c is integral, we're done!

$O(E^2 \log V \log |f^*|)$ time

(integer capacities)

"weakly polynomial time"

Ek 2) Choose augmenting path with smallest # edges.

Use BFS in $O(E)$ time per iteration.

f_i : flow after i iterations

$G_i := G_{f_i}$. $\left(f_0 : \underset{f(u \to v) = 0}{} \ \& \ G_0 = G\right)$

$\text{level}_i(v)$: unweighted distance from s to v in G_i

Lemma: $\text{level}_i(v) \geq \text{level}_{i-1}(v)$

$\text{level}_i(s) = 0 = \text{level}_{i-1}(s)$ ✓

If v cannot be reached,

$\text{level}_i(v) = \infty \geq \text{level}_{i-1}(v)$ ✓

Let $s \to \cdots \to u \to v$ be shortest in G_i.

$$\text{level}_i(v) = \text{level}_i(u) + 1$$
$$\geq \text{level}_{i-1}(u) + 1$$

If $u \to v$ is in G_{i-1},

$$\text{level}_{i-1}(u) + 1 \geq \text{level}_{i-1}(v)$$

O.W. $u \to v$ not in G_{i-1} $\Rightarrow$

we pushed along $v \to u$

$\Rightarrow$ $v \to u$ was on shortest s-t path

$$\Rightarrow \text{level}_{i-1}(u) + 1 > \text{level}_{i-1}(u) - 1$$
$$= \text{level}_{i-1}(v)$$

Either way

$$\text{level}_i(v) \geq \text{level}_{i-1}(u) + 1 \geq \text{level}_{i-1}(v)$$

Lemma: Any edge $u \to v$ leaves residual graph at most $|V|/2$.

Suppose $u \to v$ in G_i & G_{j+1} but not in $G_{i+1}, \ldots, G_j$ for some $j > i$.

$u \to v$ in ith augmenting path
$\Rightarrow \text{level}_i(v) = \text{level}_i(u) + 1$

$v \to u$ in jth augmenting path, so $\text{level}_j(u) = \text{level}_j(v) + 1$

$$\Rightarrow \text{level}_j(u) = \text{level}_j(v) + 1$$
$$\geq \text{level}_i(v) + 1 = \text{level}_i(u) + 2$$

$\Rightarrow$ it leaves & returns $\leq |V|/2$ times

$O(V)$ saturations of $O(E)$ edges

$\Rightarrow O(VE)$ iterations

$\Rightarrow O(VE^2)$ time

(for real capacities ☺)

Dinitz ['70] : $O(V^2E)$ time

...

Orlin [2012]: $O(VE)$ time

cite this in homework or exams