# CS 7301.003 Homework 1

Due Friday September 18th on eLearning

September 4, 2020

Please answer the following **3** questions, some of which have multiple parts. These questions are slight modifications of those from Erickson's Computational Topology lecture notes.

# Some important homework policies

- Groups of one or two students may work together. They should submit a single copy of their assignment using one of their eLearning accounts. Everybody in the group will receive the same grade.
- Each group must write their solutions in their own words. Clearly print your name(s), the homework number (Homework 1), and the problem number at the top of every page in case we print anything. Start each numbered homework problem on a new page.
- Unless the problem states otherwise, you must justify (prove) (argue) that your solution is correct.
- Please consider using LaTeX to typeset your solutions. Any illegible solutions will be considered incorrect. There is a template provided on the course website to help get you started.
- If you use outside sources or write solutions in close collaboration with others outside your group, then you may cite that source or person and still receive full credit for the solution. Material from the lecture, the textbook, lecture notes, or prerequisite courses need not be cited. Failure to cite other sources or failure to provide solutions in your own words, even if quoting a source, is considered an act of academic dishonesty.
- The homework is assigned to give ***you*** the opportunity to learn where your understanding is lacking and to practice what is taught in class. Its primary purpose *is not* for Kyle to grade how well you paid attention in class. Read through the questions early. Do not expect to know the answers right away. Questions are not necessarily given in order of difficulty. *Please, please, please* attend office hours or email Kyle so he can help you better understand the questions and class material.
- You may assume that reasonable operations involving a constant number of objects of constant complexity may be done in $O(1)$ time. Clearly state your assumptions if they are not something we already used in lecture.

See https://personal.utdallas.edu/~kyle.fox/courses/cs7301.003.20f/about.shtml and https://personal.utdallas.edu/~kyle.fox/courses/cs7301.003.20f/writing.shtml for more detailed policies. If you have any questions about these policies, please do not hesitate to ask during lecture, in office hours, or through email.

1. (a) Truthfully write the phrase ***"I have read and understand the course policies."***

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The proof of the Dehn-Schönflies theorem produces, for any simple polygon $P$, a ***piecewise-linear*** or ***PL*** homeomorphism $\phi$ from the plane to itself that maps $P$ to a triangle. That is, there is a triangulation $\Delta$ of the plane (or more formally, of a very large rectangle) such that the restriction of $\phi$ to any triangle in $\Delta$ is affine. The ***complexity*** of $\phi$ is the minimum number of triangles in such a triangulation.

(b) Prove that the composition of two PL homeomorphisms of the plane is another PL homeomorphism.

(c) Suppose $\phi$ is a PL homeomorphism with complexity $x$ and $\psi$ is a PL homeomorphism with complexity $y$. What can you say about the complexity of the PL homeomorphism $\psi \circ \phi$?

(d) Prove that for any simple $n$-gon $P$, there is a piecewise-linear homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ with complexity $O(n)$ that maps the polygon $P$ to a triangle. *[Hint: Consider the proof of Theorem 1.10 in Erickson's notes.]*

▷▷▷▷▷ **⟨⟨Erickson's notes are no longer available, so I'll just give everybody 2 points for this part. Feel free to answer if you'd like, but I won't grade it.⟩⟩**

(e) Prove that for any two simple $n$-gons $P$ and $Q$, there is a piecewise-linear homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ with complexity $O(n^2)$ such that $\phi(P) = Q$. *[Hint: Use the previous parts.]*

2. (a) Prove that every connected plane graph has either a vertex with degree at most 3 or a face with degree at most 3.

(b) A graph $G$ is ***bipartite*** if its vertices can be partitioned into disjoint subsets $L$ and $R$ such that every edge has one endpoint in $L$ and one endpoint in $R$. Prove that every simple bipartite planar graph has at most $2n - 4$ edges.

3. Let $G$ be an arbitrary plane graph, let $T$ be an arbitrary spanning tree of $G$, and let $e$ be an arbitrary edge of $T$. Color the vertices in one component of $T \setminus e$ red and the vertices in the other component blue. Prove that any face of $G$ is incident to either zero or two edges that have one red endpoint and one blue endpoint.