(a) Truthfully write the phrase “I have read and understand the course policies.”

Solution: I have read and understand the course policies.

(b) Prove that the composition of two PL homeomorphisms of the plane is another PL homeomorphism.

Solution: Let $\phi$ and $\psi$ be two PL homeomorphisms from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let $\Delta_{\phi_1}$ and $\Delta_{\phi_2}$ be two combinatorially identical triangulations of the plane such that $\phi$ maps each triangle of $\Delta_{\phi_1}$ affinely to its corresponding triangle of $\Delta_{\phi_2}$. Define $\Delta_{\psi_1}$ and $\Delta_{\psi_2}$ similarly.

Let $\Box$ be the planar subdivision where each face is a maximal subset of points within a unique pair of triangles, one each from $\Delta_{\phi_2}$ and $\Delta_{\psi_1}$. Let $\Delta_2$ be an arbitrary (frugal) triangulation of $\Box$. Let $\Delta_1$ be the collection of triangles obtained by restricting $\phi^{-1}$ to each triangle of $\Delta_2$, and let $\Delta_3$ be the triangles obtained by restricting $\psi$ to each triangle of $\Delta_2$. Compositions of linear functions are linear, and composition of homeomorphisms are homeomorphisms. Therefore, $\psi \circ \phi$ applies an invertable linear map to any triangle of $\Delta_1$ to a triangle of $\Delta_2$ to a triangle of $\Delta_3$. Function $\psi \circ \phi$ is a PL homeomorphism.

(c) Suppose $\phi$ is a PL homeomorphism with complexity $x$ and $\psi$ is a PL homeomorphism with complexity $y$. What can you say about the complexity of the PL homeomorphism $\psi \circ \phi$?

Solution: Let $\Delta_{\phi_1}$ and $\Delta_{\psi_1}$ be minimum size triangulations as defined in part (b). The complexity of $\psi \circ \phi$ is at most the size of $\Delta_2$. The intersection of two triangles in $\mathbb{R}^2$ has at most 6 edges, so $\Box$ and therefore $\Delta_2$ have complexity at most $O(x \cdot y)$.

This complexity is asymptotically tight in the worst case. Consider the following triangulation consisting of a stack of $x$ horizontally aligned skinny triangles.

Let $\phi$ be a PL homeomorphism that reverses the bottom-to-top order of the blue triangles with their horizontal side to the left. Let $\psi$ be a similarly defined homeomorphism that reverses the left-to-right order of some vertically aligned triangles with their vertical side on the bottom. If $\phi$ has complexity $x$ and $\psi$ has complexity $y$, their composition will shift around $\Omega(x \cdot y)$ separate regions of the plane and have complexity $\Omega(x \cdot y)$. 

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(d) Prove that for any simple \( n \)-gon \( P \), there is a piecewise-linear homeomorphism \( \phi : \mathbb{R}^2 \to \mathbb{R}^2 \) with complexity \( O(n) \) that maps the polygon \( P \) to a triangle.

**Solution:** (Sketch) There is a PL homeomorphism of complexity 1 from any triangle to any other triangle, so it suffices to show a homeomorphism from \( P \) to a particularly convenient triangle. Let \( T \) be the triangle with vertices at \((0, 0), (n-2, 0), \) and \(((n-2)/2, (n-2)^2)\). Let \( P \) have vertices \((p_0, p_1, \ldots, p_{n-1})\). Without loss of generality, let \( \Delta \) be any frugal triangulation of \( P \) that does not include a diagonal incident to \( p_{n-1} \) (perhaps \( p_{n-1} \) is the middle vertex of an ear). For each diagonal \( p_ip_j \) in \( \Delta \) with \( i < j \), subdivide \( T \) by adding a path from \((i, 0)\) to \(((i+j)/2, (j-i)^2)\) to \((j, 0)\). We can easily verify that these paths are interiorly disjoint, and \( T \) now has \( O(n) \) faces. We can then triangulate each face to find a (non-frugal) triangulation of \( T \) of size \( O(n) \). For each edge added during this triangulation of \( T \), we add a corresponding edge between diagonals of \( P \). Now \( P \) and \( T \) have combinatorially identical triangulations. We can find combinatorially identical triangulations of the outside of \( P \) and \( T \) in a similar manner. Let \( \phi \) map each triangle around \( P \) to its corresponding triangle around \( T \). PL homeomorphism \( \phi \) has complexity \( O(n) \). 

(e) Prove that for any two simple \( n \)-gons \( P \) and \( Q \), there is a piecewise-linear homeomorphism \( \phi : \mathbb{R}^2 \to \mathbb{R}^2 \) with complexity \( O(n^2) \) such that \( \phi(P) = Q \).

**Solution:** Let \( \phi_1 : \mathbb{R}^2 \to \mathbb{R}^2 \) be a PL homeomorphism of complexity \( O(n) \) from \( P \) to a triangle \( T \). Let \( \phi_2 : \mathbb{R}^2 \to \mathbb{R}^2 \) be a PL homeomorphism of complexity \( O(n) \) from \( Q \) to \( T \). Part (d) guarantees these homeomorphisms exist. Let \( \phi = \phi_2^{-1} \circ \phi_1 \). PL homeomorphism \( \phi \) has complexity \( O(n^2) \) according to part (c).
(a) Prove that every connected plane graph has either a vertex with degree at most 3 or a face with degree at most 3.

**Solution:** Let $G = (V, E)$ be a connected plane graph where every vertex has degree at least 4. We have $2m \geq 4n$, implying $n \leq m/2$. By Euler’s formula, $f - m/2 \geq 2$, implying $2m \leq 4f - 8$. The average face degree is strictly less than 4, so there exists at least one face with degree at most 3.

(b) Prove that every simple bipartite planar graph has at most $2n - 4$ edges.

**Solution:** Let $G = (V, E)$ be a simple bipartite plane graph. The statement is easily shown incorrect for $n \leq 2$, so we’ll assume $n \geq 3$. Graph $G$ has no faces of degree 1 or 2, because it is simple. It has no faces of degree 3, because it is bipartite. Therefore, every face has degree at least 4. We have $2m \geq 4f$, implying $f \leq m/2$. By Euler’s formula, $n - m/2 \geq 2$, implying $m \leq 2n - 4$. 


Let $G$ be an arbitrary plane graph, let $T$ be an arbitrary spanning tree of $G$, and let $e$ be an arbitrary edge of $T$. Color the vertices in one component of $T \setminus e$ red and the vertices in the other component blue. Prove that any face of $G$ is incident to either zero or two edges that have one red endpoint and one blue endpoint.

**Solution:** Let $C$ be the set of edges with one blue endpoint and one red endpoint. Set $C$ is an edge cut, because it leaves the subset of red vertices. Further, $C$ is a bond, because removing any one edge from $C$ would reconnect the two components of $T$. Dual edge set $C^*$ is therefore a cycle. Every dual vertex is incident to zero or two edges of $C^*$, so every primal face is incident to zero or two edges of $C$. \qed