CS 7301.003 Homework 2

Due Friday October 9th on eLearning

September 25, 2020

Please answer the following **3** questions, some of which have multiple parts. These questions are slight modifications of those from Erickson's Computational Topology courses.

1. Let $\mathcal{P}$ be a polygon with h holes, specified by polygons $P_0, P_1, \ldots, P_h$. For this problem, we define an ***arc*** in $\mathcal{P}$ to be any path in $\mathcal{P}$ whose endpoints lie on the outer polygon P_0. Two arcs α and β are ***homotopic relative to*** $\mathcal{P}$, or just ***relatively homotopic***, if there is a continuous function $h : [0,1] \times [0,1] \to \mathcal{P}$ satisfying four conditions:

   - $h(s,0) = \alpha(s)$ and $h(s,1) = \beta(s)$ for all $s \in [0,1]$.
   - $h(0,t) \in P_0$ and $h(1,t) \in P_0$ for all $t \in [0,1]$.

   Informally, a relative homotopy deforms one arc into another through a continuous sequence of arcs.

   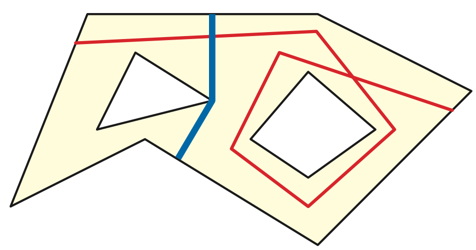

   Figure 1. The blue (thick) arc is relatively homotopic to the red (thin) arc.

   Describe an algorithm to determine whether two given *polygonal* arcs α and β are relatively homotopic. You should attempt to justify your algorithm's correctness, but you do not need to give a formal proof.

   ▷▷▷▷▷ **⟨⟨The original hint given below is suggesting a solution to a related but different problem of determining whether two paths in a sphere with h holes are homotopic. A correct polynomial time algorithm for *either* the original problem of testing relative homotopy or the alternate problem of testing homotopy on a sphere with holes is worth full credit.⟩⟩**

   [Relative homotopy hint: How does a homotopy moving the beginning or ending of an arc close to P_0 affect a path's crossing sequence? How can you modify the regular homotopy algorithm to account for these kinds of changes?]

   [Regular homotopy on sphere hint: Recall the $O(n + h \log h + hk)$ time algorithm for regular homotopy where we begin by triangulating a large rectangle enclosing h sentinal points, each lying within a hole of $\mathcal{P}$. What should you triangulate instead to cleanly handle homotopy on a sphere with holes?]

2. When we think about algorithms for planar graphs, it is sometimes useful to make assumptions about the degrees of vertices and/or the degrees of faces. Let G be a simple undirected plane graph with weighted edges.

   - We can assume without loss of generality that every vertex of G has degree 3 by expanding each higher-degree vertex into a tree of degree-3 vertices. Moreover, we can preserve shortest-path distances by giving these new edges weight (length) zero, and we can preserve maximum flow values by giving the new edges infinite weight (capacity).
   - We can assume without loss of generality that every face of G has degree 3 by inserting diagonals into any higher-degree face. Moreover, we can preserve shortest-path distances by giving these new edges infinite weight (length), and we can preserve maximum flow values by giving the new edges weight (capacity) zero.

   Unfortunately, expanding vertices increases face degrees, and triangulating faces increases vertex degrees. What if we need both vertices and faces to have small degree?

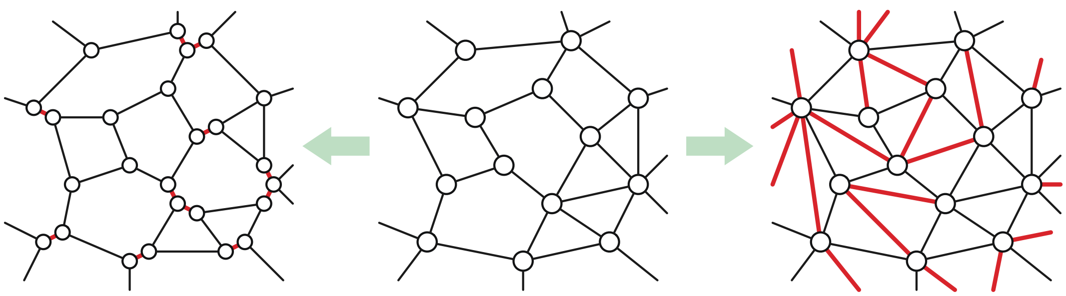

Figure 2. Center to left: expanding vertices. Center to right: triangulating faces.

   Prove that for some constants Δ, Δ^*, and c, any simple n-vertex plane graph G can be modified by inserting subgraphs or expanding vertices into a new simple planar graph $\tilde{G}$ with cn vertices, such that each vertex has degree at most Δ and each face has degree at most Δ^*. More simply: Argue that without loss of generality, simple plane graphs have bounded vertex degrees *and* bounded face degrees. You do not need to give exact values for these constants, but do argue that they exist.

   [Hint: Start by expanding all high degree vertices. Then, for each large face, subdivide it by inserting a single smaller face (not necessarily of constant size) surrounded by multiple faces that are of constant size each, and recurse.]

3. For the purpose of this question, define a ***directed graph*** as an abstract graph where each edge is assigned a direction from one end point to the other. Specifically, for any edge e consisting of darts e^+ and e^-, we consider $tail(e) := tail(e^+)$ and $head(e) := head(e^+)$. Therefore, in a directed *plane* graph, the dual of a directed edge $e = tail(e){\to}head(e)$ is $e^* = left(e)^*{\to}right(e)^*$. A directed graph is ***acyclic*** if it contains no directed cycles, and it is ***strongly connected*** if there is a directed path between every pair of vertices u and v. A ***strong component*** is a maximal strongly connected subgraph.

   (a) Prove that a directed plane graph G is acyclic if and only if the dual graph G^* is strongly connected.

(b) Call an edge of a directed graph G ***internal*** if its endpoints lie in the same strong component of G and ***external*** otherwise. Prove that an edge e in a directed plane graph G is internal if and only if the corresponding dual edge e^* of the dual graph G^* is external.