Please answer the following 2 questions, some of which have multiple parts. These questions are slight modifications of those from Erickson’s Computational Topology courses.

1. (a) Let $\Sigma$ be a combinatorial 2-manifold, where each corner $x$ of each face of $\Sigma$ is assigned a positive real number $\angle x$, called the angle at $x$. Let corners($v$) or corners($f$) denote the set of corners incident to a vertex $v$ or a face $f$, respectively. We define the curvature of each vertex and face as follows:¹

$$\kappa(v) := 1 - \sum_{x \in \text{corners}(v)} \angle x$$

$$\kappa(f) := 1 - \sum_{x \in \text{corners}(f)} (1/2 - \angle x).$$

Prove the combinatorial Gauß-Bonnet theorem:

$$\sum_{\text{vertex } v \text{ of } \Sigma} \kappa(v) + \sum_{\text{face } f \text{ of } \Sigma} \kappa(f) = \chi(\Sigma).$$

(b) Suppose every face of $\Sigma$ is a triangle. Prove the following special case of the combinatorial Gauß-Bonnet theorem:

$$\sum_{\text{vertex } v \text{ of } \Sigma} (6 - |\text{corners}(v)|) = 6 \chi(\Sigma).$$

(c) Now suppose $\Sigma$ has boundary components. Let $\chi(v)$ denote the number of edges incident to $v$, counting loops twice, minus the number of corners incident to $v$. We now redefine the curvature of a vertex $v$ as follows:

$$\kappa(v) := 1 - \frac{\chi(v)}{2} - \sum_{x \in \text{corners}(v)} \angle x.$$

Prove that the combinatorial Gauß-Bonnet theorem holds in this more general setting.

(d) Suppose the surface $\Sigma'$ is homeomorphic to a disk, and every face and interior vertex of $\Sigma'$ has curvature at most 0. Prove that at least three boundary vertices of $\Sigma'$ have strictly positive curvature.

¹The definitions use individual passes around a circle as the unit of angular measurements.
2. Let $G$ be a cellurally embedded (i.e. every face is a disk) graph on a surface $\Sigma$ with boundary. Recall, a cut graph is a subgraph $H$ of $G$ such that the closure of $\Sigma \setminus H$ is a disk. A cut graph is minimal if no proper subgraph is a cut graph. For example, a minimal cut graph of an annulus is a path from one boundary to the other.

A pair of pants is a sphere minus three open disks. Let $G$ be a graph with non-negatively weighted edges, cellulary embedded in a pair of pants $\Sigma$. Describe an algorithm to find the minimum-length cut graph in $G$ in $O(n \log n)$ time. [Hint: What does a minimal cut graph of a pair of pants look like?]