CS 7301.003 Homework 3

Due Friday November 20th on eLearning

November 2, 2020

Please answer the following **2** questions, some of which have multiple parts. These questions are slight modifications of those from Erickson's Computational Topology courses.

1. (a) Let Σ be a combinatorial 2-manifold, where each corner x of each face of Σ is assigned a positive real number $\angle x$, called the ***angle*** at x. Let corners(v) or corners(f) denote the set of corners incident to a vertex v or a face f, respectively. We define the ***curvature*** of each vertex and face as follows:[1]

$$\kappa(v) := 1 - \sum_{x \in \text{corners}(v)} \angle x \qquad \kappa(f) := 1 - \sum_{x \in \text{corners}(f)} (1/2 - \angle x).$$

Prove the ***combinatorial Gauß-Bonnet theorem***:

$$\sum_{\text{vertex } v \text{ of } \Sigma} \kappa(v) + \sum_{\text{face } f \text{ of } \Sigma} \kappa(f) = \chi(\Sigma).$$

(b) Suppose every face of Σ is a triangle. Prove the following special case of the combinatorial Gauß-Bonnet theorem:

$$\sum_{\text{vertex } v \text{ of } \Sigma} (6 - |\text{corners}(v)|) = 6\chi(\Sigma).$$

(c) Now suppose Σ has boundary components. Let $\chi(v)$ denote the number of edges incident to v, counting loops twice, minus the number of corners incident to v. We now redefine the curvature of a vertex v as follows:

$$\kappa(v) := 1 - \frac{\chi(v)}{2} - \sum_{x \in \text{corners}(v)} \angle x.$$

Prove that the combinatorial Gauß-Bonnet theorem holds in this more general setting.

(d) Suppose the surface Σ' is homeomorphic to a disk, and every face and *interior* vertex of Σ' has curvature at most 0. Prove that at least three boundary vertices of Σ' have strictly positive curvature.

[1]The definitions use individual passes around a *circle* as the unit of angular measurements.

2. Let G be a cellularly embedded (i.e. every face is a disk) graph on a surface Σ *with boundary*. Recall, a ***cut graph*** is a subgraph H of G such that the closure of $\Sigma \setminus H$ is a disk. A cut graph is minimal if no proper subgraph is a cut graph. For example, a minimal cut graph of an annulus is a path from one boundary to the other.

   A ***pair of pants*** is a sphere minus three open disks. Let G be a graph with non-negatively weighted edges, cellulary embedded in a pair of pants Σ. Describe an algorithm to find the minimum-length cut graph in G in $O(n \log n)$ time. *[Hint: What does a minimal cut graph of a pair of pants look like?]*