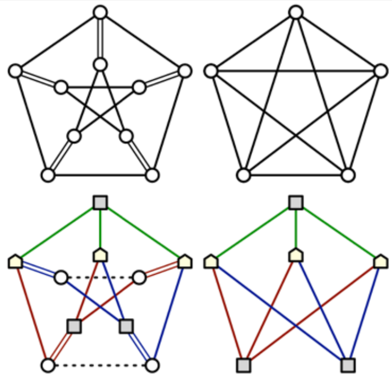
CS 7301.003.20F Lecture 18—October 19, 2020

Main topics are #graph_minors .

Graph Minors

- Recall a *minor* of a graph G is obtained by contracting some edges of a subgraph of G. A *proper minor* is any minor other than G itself. For example, K_5 and K_3,3 are minors of the Peterson graph.

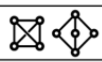


Both K_5 and $K_{3,3}$ are minors of the Peterson graph.
Doubled edges are contracted; dashed edges are deleted.

- The following well-known theorem is from Wagner's 1935 PhD thesis:
- Theorem: A graph G is planar if and only if K_5 and K_3,3 are not minors of G.
- Kuratowski showed a similar theorem about subdivisions of K_5 and K_3,3.
- In the same thesis, Wagner described the family of all graphs that do not have K_5 as a minor.
- More generally, we say a graph H is a *forbidden minor* for a set F of graphs if H is not a minor of any graph in F. A forbidden minor H of F is minimal if not proper minor H is also forbidden from F.
- If graph family F is defined by having least one forbidden minor, then it is *minor-closed*: every minor of a graph in F is also in F.
- Conversely, if a minor-closed graph family F excludes a graph H, then it also excludes any graph for which H is a minor. Therefore, every minor-closed family of graphs, except for the family of all graphs, has at least one forbidden minor.
- In the mid-1980s, Robertson and Seymour began publishing several papers totalling several hundred pages concerning graph minors. In them, they proved one of the deepest theorems of combinatorics:
- The Graph Minor Theorem: In any infinite set of graphs, at least one graph is a proper minor of another.
- The proof of this theorem is well beyond the scope of this course, but we can discuss some partial results that came from Robertson and Seymour's work.

Minor-Closed Families

- Many natural families of graphs are minor-closed. For example, the family graphs embeddable on any fixed 2-manifold (and in particular, the plane).
- The family of trees is also minor-closed as are families of graphs with low *treewidth*, which we'll define later.
- An immediate corollary of the Graph Minor Theorem gives us a less natural, but perhaps more algorithmically useful way to describe any minor-closed family.
- Theorem: A family of graphs is minor-closed if and only if it has a finite number of minimal forbidden minors.
    - If the forbidden set were infinite, then at least one member is not minimal.
- So all minor-closed families have Kuratowski-Wagner style theorems.
- The set of minimal forbidden minors for a family is sometimes called its *obstruction set*. Here's a list of obstruction sets for some important families.

Family	**Obstruction set**	
treewidth 1 (forests)		
treewidth 2		
outerplanar		
planar		[43]
treewidth 3		[6, 34]
linklessly embeddable	7 graphs	[32, 33, 31]
projective-planar	35 graphs	[16, 3]

- As far as a know, the full list for even toroidal graphs is not explicitly known. In fact, there cannot exist any algorithm that takes an arbitrary minor-closed family and computes its obstruction set.
- However, we can determine if a fixed graph is the minor of another.
- Theorem [R & S]: For any fixed graph H, there is an algorithm to determine whether a given n-node graph has H as a minor in O(n^3) time.
- In particular, that theorem and the previous one imply the existence of an algorithm for testing membership in any minor-closed family. But we only knows it exists, not what the algorithm for arbitrary H look like.
- And the guaranteed dependence on |H| is really really ridiculously huge.
- Minor closed families share an important feature that we've taken advantage of when discussing surface embedded graphs.
- Theorem [Kostochka '82, '84; Thomason '84, '01]: For every fixed graph H, any n-vertex H-minor-free graph has O(n) edges.
- In particular, n-vertex K_k-minor-free graphs have O(nk sqrt(log k)) edges.

k-Trees, Treewidth, and Grids

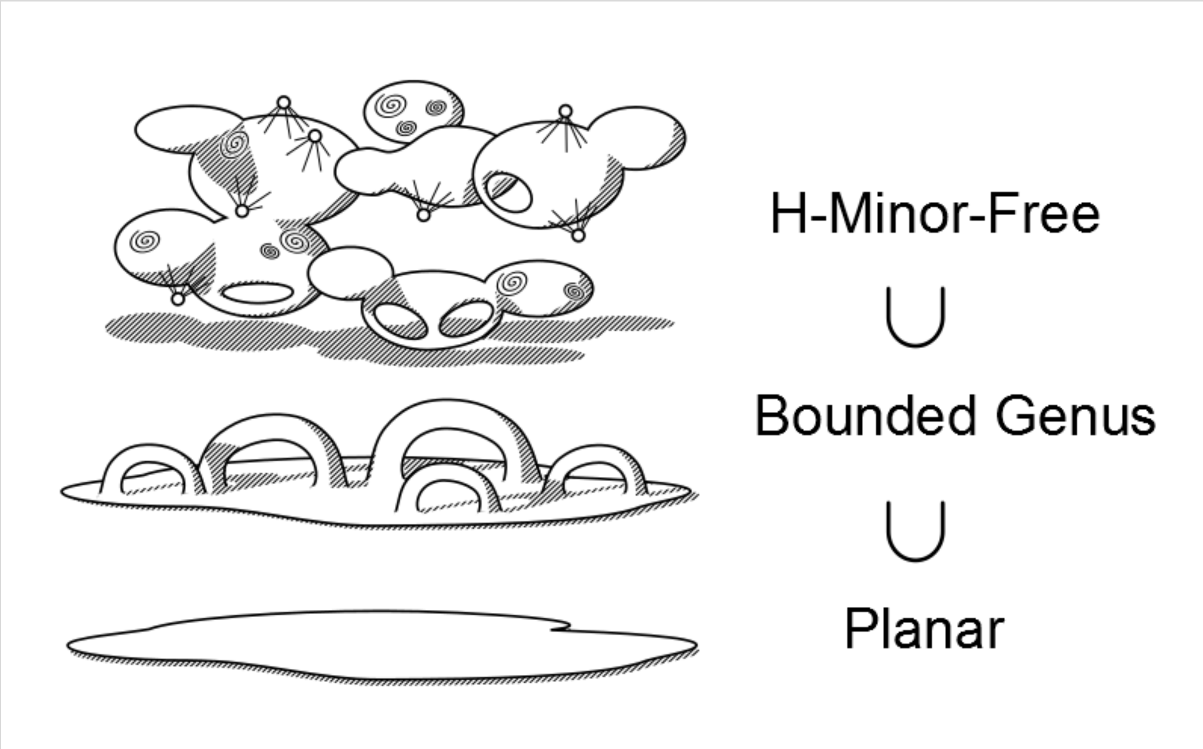
- There's a particularly important minor closed graph family that we haven't discussed yet.
- First off, a *k-tree* is either
  - The complete graph $K_{k + 1}$.
  - Or the addition of a single vertex v of degree k to a k-tree such that v and its k neighbors form a clique
- The 1-trees are the trees with at least one edge.
- A *partial k-tree* is a subgraph of a k-tree. We say a graph has *treewidth* k if and only if it is a partial k tree but not a partial (k - 1) tree.
- Partial 1-trees are the forests. Partial 2-trees are the series parallel graphs.
- The graphs of treewidth at most k form a minor free family.
- As we'll see next time, graphs of treewidth at most k often have efficient dynamic programming algorithms even for problems that are NP-hard in general graphs, and in turn, that leads to many good approximation algorithms for surface embedded graphs.
- In short, small treewidth implies existence of a balanced separator of about the same size.
- Treewidth turns out to be an important concept when discussing other minor-free families as well.
- Theorem [Alon et al. '90, Plotkin et al. '94]: For every fixed graph H, any n-vertex-minor-free graph has treewidth $O(\sqrt{n})$.
- In particular, n-vertex K_k-minor-free graphs have treewidth $O(k \sqrt{n} \sqrt{\min \{k, \log n\}})$.
- In particular, in particular, graphs embeddable on genus g surfaces have balanced separators of size $O(\sqrt{ng})$.
- The worst case for treewidth is the grid graph. The *r x r grid* is a graph G = (V, E) where $V = \{1, 2, \ldots, r\} \times \{1, 2, \ldots, r\}$ and two pairs (i, j) and (i', j') share an edge if and only if $|i - i'| + |j - j'| = 1$.
- The r x r grid has treewidth exactly r. But strangely, grids are essentially the only kinds of graphs with large treewidth.
- Theorem [R & S]: There is a function $f : N \to N$ such that for every integer r, every graph of treewidth f(r) has an r x r grid minor.
- Originally, the best known upper bound for this function was $2^{O(r^5)}$, but recently Chekuri and Chuzhoy ['16] decreased it to $O(r^{36} \text{ polylog } r)$.
- A surprising corollary of the previous theorem is the following:
- Theorem: A minor-closed family of graphs has bounded treewidth if and only if it excludes at least one planar graph.
- Proof:
  - If a family contains all planar graphs, it contains all grids and therefore it does not have bounded treewidth.
  - On the other hand, if a family F excludes an r-vertex planar graph, then it excludes an

O(r) x O(r) grid. Therefore, it has treewidth at most O(f(r)).

- Notice one weird consequence of this theorem. The worst-case treewidth of an n-vertex graph in a minor closed family F is either Theta(sqrt{n}) (if F includes all planar graphs) or Theta(1) (if F excludes at least one planar graph).
- Finally, Demaine and Hajiaghayi ['08] showed an even stronger grid theorem for families forbidding a fixed minor.
- Theorem: For any fixed graph H, every H-minor-free graph of treewidth w has an Omega(w) x Omega(w) grid as a minor.
- This theorem turns out to be important for certain fixed parameter tractable algorithms.

Decomposition Theorem

- One final part of Robertson and Seymour's theorem with useful consequences in algorithms is a theorem for decomposing graphs in a minor-free family into certain better understood structures.
- First, a *k-clique sum* of two graphs G and H is a graph obtained by identifying a clique of at most k vertices in G with a clique of the same size in H and then maybe removing some edges from the shared clique.
- By definition, every k-tree is the clique-sum of (k + 1)-cliques and no deletions. Every graph of treewidth k is a k-clique-sum of graphs with at most k + 1 vertices.
- A graph H is a k-apex graph of a graph G if G = H \ A for some subset A of at most k vertices, called apices.
- Finally, a *vortex* is a graph with small treewidth that is glued into a face of an embedded graph in a particular way; I can't give you details until we've discussed treewidth more. There's a notion of pathwidth that is similar to treewidth that applies to vortices.
- We say a graph G is *cleanly k-almost-embeddable* on a surface Sigma if G can be written as the union of k + 1 graphs G_0 cup G_1 cup … G_k such that
    - G_0 has an embedding on Sigma
    - and the graphs G_1, G_2, …, G_k are pairwise disjoint vortices embedded into the faces of G_0 in a particular way that I can't really define without explaining pathwidth
- Here's the main theorem:
- Theorem [Robertson and Seymour]: For any graph H, there is an integer k = k(H) such that any H-minor-free graph is a k-clique sum of a finite number of k-apex graphs of cleanly k-almost-embeddable graphs on an orientable surface of genus k.
- Here's a helpful illustration by Felix Reidl:



- Demaine et al. ['05] describe an algorithm for any fixed forbidden minor H that computes this decomposition. The decomposition can be used in various algorithms using a kind of "divide-and-conquer" approach.