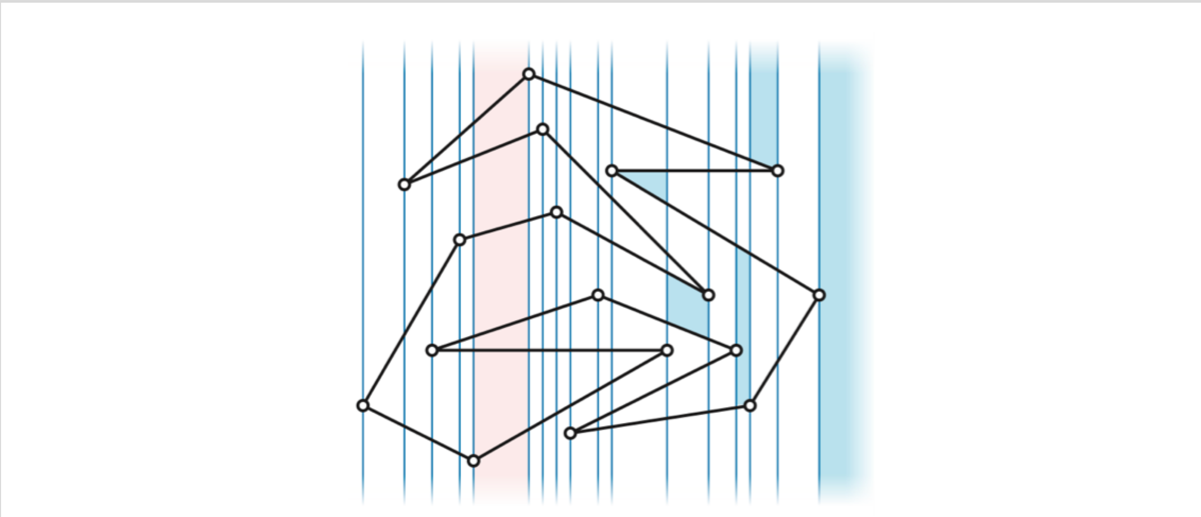
# CS 7301.003.20F Lecture 2—August 19, 2020

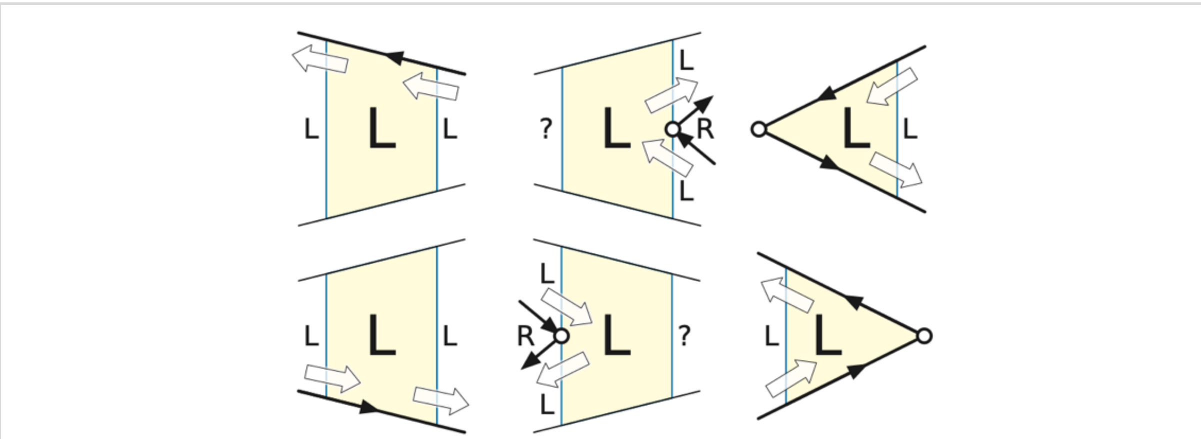
Main topics are #introduction , #administrivia , #triangulations , #polygon_homeomorphisms , and #planar_straight-line_graphs .

## The Jordan Polygon Theorem

- Last time, we finished by stating the Jordan Polygon Theorem: The complement R^2 \ P of any simple polygonal chain P in the plane has exactly two components.
- I started sketching a proof that follows arguments by Schönflies (1896).
- Let ell_i be the vertical line through p_i. These vertical slabs. The edges of P further divide the slabs into trapezoids.



- Today, I'll argue that R^2 \ P has at most two components.
  - Imagine each edge p_i p_{i + 1} as directed from p_i to p_{i + 1}. We'll label a trapezoid as *left* if at least one of the following is true:
    - The floor is directed left to right.
    - The ceiling is directed right to left.
    - The right wall contains a vertex p_i, and p_{i - 1} p_i is below p_i p_{i + 1}.
    - The left wall contains a vertex p_i, and p_{i - 1} p_i is above p_i p_{i + 1}.




  - *Right* trapezoids have a symmetric definition. Every trapezoid has at least one of the

    four things mentioned in the bullets, so every trapezoid must be left, right, or (as far as we can tell so far) both.
    - Now, imagine creating a sequence of left trapezoids as we walk along the polygon from p_0 to p_n = p_0. As we traverse an edge p_i p_{i + 1} directed right (resp. left), we add trapezoids just above (resp. below) the edge in order from left to right (resp. right to left). When we traverse p_i whose neighbors are both right of p_i with p_{i - 1} p_i above p_i p_{i + 1}, we add the trapezoid just left of p_i to the sequence. When we traverse p_i whose neighbors are both left of p_i with p_{i - 1} p_i is below p_i p_{i + 1}, we add the trapezoid just to its right to the sequence.
    - Every left trapezoid appears at least once in the sequence, and adjacent members of the sequence share a wall, so the union of the left trapezoids is connected.
    - Similarly, the union of the right trapezoids are connected.
- On Monday, I argued  R^2 \ P has at least two components. We labeled each trapezoid *even* or *odd* to match the parity of the number of polygon edges. All trapezoids in a single component have the same parity, and there's at least one trapezoid of each parity, so there are at least two components.
- Notice how trapezoids containing infinitely high points "outside" P are even. Therefore, the one outside component has all the even trapezoids and the one inside component has all the odd ones. So, we can test if a given point lies inside, outside, or on P by asking if it lies in an odd trapezoid, in an even trapezoid, or on an edge. To do so, we just loop through in O(n) time, checking which edges lie above the point.

```
PointInPolygon(P[0..n−1], q):
    sign ← −1
    P[n] ← P[0]
    for i ← 0 to n − 1
        sign ← sign · OnOrBelow(q, P[i], P[i + 1])
    return sign
```

```
OnOrBelow(q, r, s):
    if r.x < s.x
        swap r ↔ s
    if (q.x < s.x) or (q.x ≥ r.x)
        return +1
    return sgn(Δ(q, r, s))
```

- The algorithm PointInPolygon returns +1, -1, or 0 to indicate point q is inside, outside, or on P.
- OnOrBelow(q, r, s) returns -1 if q lies below segment rs, 0 if q lies on rs, or +1 otherwise.
- It uses a routine sgn which tells you if the triple (q, r, s) of points is oriented counterclockwise. You may recognize such a test if you took computational geometry.

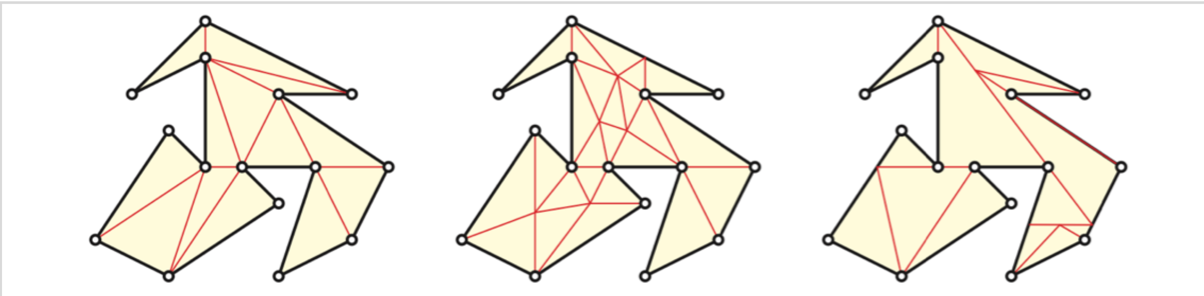
## Triangulations

- So now we've discussed polygonal chains and seen that simple polygonal chains always split the plane into two parts. Today, we're going to discuss a useful tool for working with polygonal chains. It will serve us well when we discuss problems on curves in a couple weeks, and it naturally leads to next week's main topic.
- When working with simple polygons, it's useful to decompose their interiors into simpler

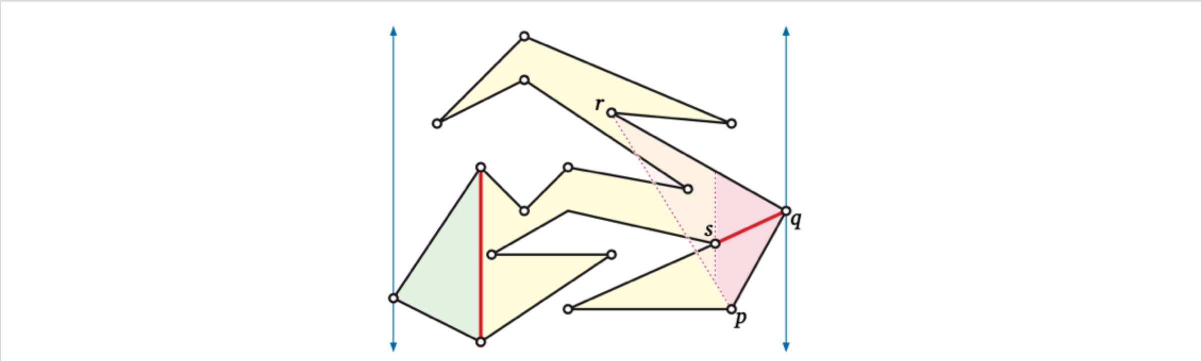
pieces using a triangulation. A *triangulation* is a triple of sets (V, E, T)

- V is a finite set of points in the plane called *vertices*
- E is a set of interior-disjoint line segments between points in V called *edges*
- T is a set of interior-disjoint triangles, called *faces*, whose vertices come from V and whose edges come from E
- every point in V is a vertex of at least one triangle
- every segment in E is an edge of at least one triangle

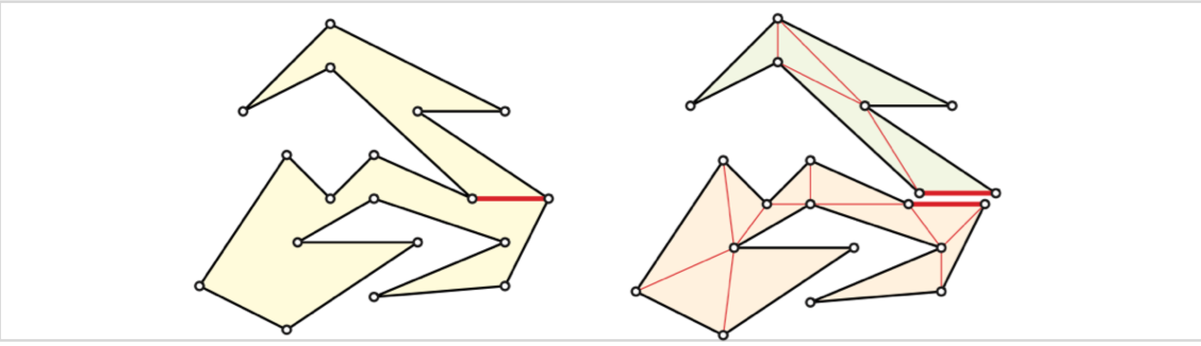
- If the union of triangles is the closure of the interior of a simple polygon P, then (V, E, T) is a *triangulation of P*
- This definition is a bit more relaxed than ones you may have seen before. It's true that every vertex of P is a vertex of the triangulation, but some edges of P may be the union of multiple vertices and edges of the triangulation
- But as you might suspect, every edge lies on exactly one or two polygons. And pairs of triangles intersect along an edge, at a single vertex, or not at all.
- A triangulation of P is *frugal* if its vertices are precisely those of P.
- A *diagonal* of simple polygon P is a line segment whose endpoints are vertices of P and otherwise it lies interior to P. So every edge of a frugal polygon is either from P or a diagonal.
- So going left to right, we have a frugal triangulation, a non-frugal triangulation, and something at isn't a triangulation at all.



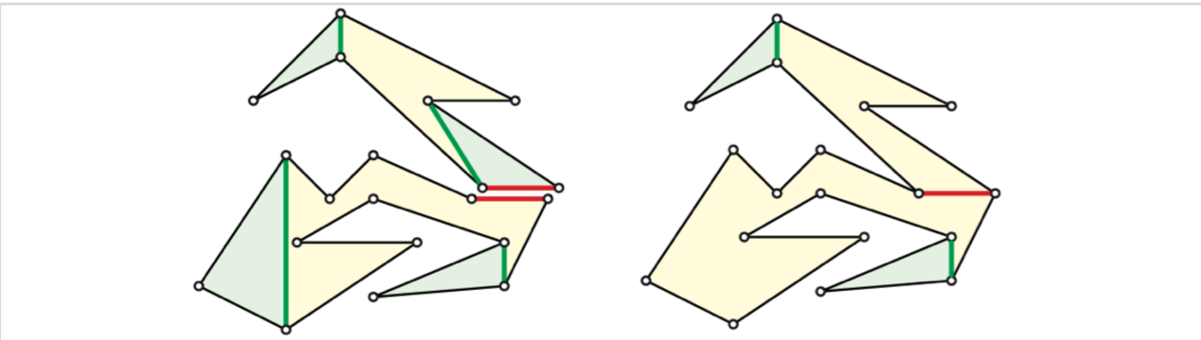
- Dehn (1899) and Lennes (1903) showed every simple polygon with at least four vertices has a diagonal.
    - In short, you take the rightmost vertex q and the surrounding vertices p and r. Either pr is otherwise disjoint from P and you can prove it is a diagonal, or it intersects P somewhere. Then there exists a rightmost vertex s in triangle pqr and have diagonal sq.




- By finding such a diagonal, cutting the polygon along it, and recursively triangulating both sides, we can prove that every simple polygon has a frugal triangulation.



- A triangle is called an *ear* if it is made from two edges of P and one diagonal. Dehn (1899) and Meisters (1975) observed that every polygon with at least four vertices has an ear.
  - In fact, they all have at least two ears. Find a diagonal pq and cut along it like before to get simple polygons P′ and P″. If P′ is a triangle then it is an ear of P. Otherwise, P′ contains two ears, and the one not on pq must be an ear of P. The second ear of P comes from P″.

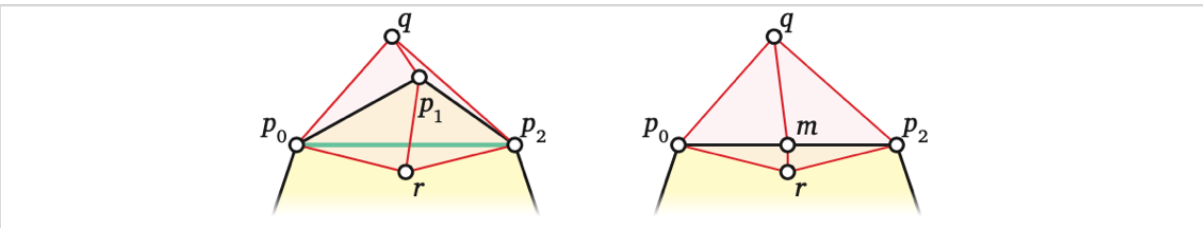


- The inductive proof mentioned earlier implies an O(n^2) time algorithm for constructing a frugal triangulation of any simple polygon: find a diagonal, cut open the polygon, and recursively triangulate both sides. If you're familiar with much computational geometry, you may already know that better algorithms exist. You can look at my notes for that class for an O(n log n) time algorithm if you're interested. There's also a much more complicated O(n) time algorithm.

## The Jordan-Schönflies Theorem

- For our last deep topic of the day, we're going to extend the Jordan curve theorem in a way that lets me introduce one of the most fundamental topics of topology.

- A *homeomorphism* is a continuous function with a continuous inverse.
- Two spaces are *homeomorphic* if there is a homeomorphism between them.
- A simple example is how any simple closed curve can be defined as any subset of the plane homeomorphic to the circle $S^1$.
- The following theorem is attributed to Schönflies (1906).
- The Jordan-Schönflies Theorem: For any simple closed curve C in the plane, there is a homeomorphism from the plane to itself that maps C to the unit circle $S^1$.
- You actually need this theorem to know that not only does $R^2 \setminus C$ have two components, but C is the boundary of both, and that the closure of the inner component is homeomorphic to the unit disk $B^2 := \{(x, y) \text{ in } R^2 : x^2 + y^2 \leq 1\}$. "Obvious" statements, but the 3D generalization is actually false!
- Again, I can't give the general proof, but I can go with a specialization to polygons, which Dehn (1899) also observed.
- The Dehn-Schönflies Polygon Theorem: For any simple polygon P, there is a homeomorphism from the plane to itself that maps P to the boundary of a triangle.
    - Like the frugal triangulation proof above, there's actually a nice little algorithm for describing this homeomorphism.
    - Let P have vertices $p_0, \ldots, p_{n-1}$. If P is a triangle, then we use the identity homeomorphism, so assume $n \geq 4$.
    - Let's suppose without loss of generality that $p_1$ is the tip of an ear, and let P' be the simple polygon $p_0\, p_2, p_3, \ldots, p_{n-1}$. Inductively, there is a homeomorphism $\phi' : R^2 \to R^2$ that maps P' to the boundary of a triangle. We're going to describe a homeomorphism $\psi : R^2 \to R^2$ mapping P to P' and then compose that one with $\phi'$.
    - Let q be a point close to $p_1$ but outside P, let m be the midpoint of $p_0\, p_2$, and let r be a point close to m in P but outside triangle $p_0\, p_1\, p_2$.




    - Let Q be the quadrilateral $p_0\, q\, p_2\, r$. We have two combinatorially isomorphic triangulations of Q, one with internal vertex $p_1$ and one with internal vertex m.
    - Let psi be the piecewise-affine map where outside Q doesn't change, $p_1$ maps to m, and we linearly extend psi across each of the four triangles of the first triangulation.
- There's a weaker version of the theorem that might be a bit more intuitive and whose proof might get you in a better mindset for algorithms.