# CS 7301.003.20F Lecture 20—October 26, 2020

Main topics are #cell_complex.

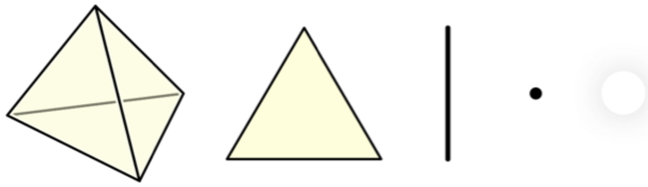
## Cell Complexes

- We've talked a lot about 2-manifolds and representing them with graph embeddings. Today we're finally going to generalize this idea to higher dimensions by talking about cell complexes.
- A *cell complex* is a decomposition of a topological space into pieces of simple topology that are glued together 'nicely' along their boundaries. There are many kinds that all have different definitions. So let's go!
- Let's start general to establish some common terminology before we swing to the other extreme.
- An *abstract simplicial complex* is a collection calX of finite sets closed under taking subsets.
- The sets in X are called *simplices*. The *vertices* of X are the singleton sets (or their elements) denoted as calX_0.
- The *dimension* of a simplex is one less than its cardinality (so vertices have dimension 0) and the *dimension* of the complex is the largest dimension of any simplex.
- Subsets of a simplex are its *faces*. A simplex is *maximal* or a *facet* if it is not a proper face of any other simplex. Finally, calX is *pure* if every facet has the same dimension.
- Some examples:
  - Power set of any finite set
  - Simple undirected graphs with no isolated vertices are pure 2-dim ASCs.

## Geometric Simplicial Complexes

- But we need something a bit more concrete to hang out hat onto.
- It turns out any abstract simplicial complex calX has a *geometric realization*, a map of calX to generic points in a sufficiently high dimensional space.
- The *convex hull* of a finite set X subset R^d is the set of all weighted averages of points in X.
  - conv(X) := {sum_{x in X} lambda_x x | sum_{x in X} lambda_x = 1 and 0 ≤ lambda_x ≤ 1 for all x in X}
- The convex hull of a finite set of points is called a *polytope*. A *proper face* of a polytope is its intersection with any hyperplane that does not intersect the interior. Any proper face of a polytope is also a polytope.
- Say k points in R^d are *affinely independent* if their convex hull is k-dimensional.

- A *k-simplex* is the convex hull of a set of k + 1 affinity independent points called its *vertices*.



A 3-simplex, a 2-simplex, a 1-simplex, a 0-simplex, and the (-1)-simplex

- A *face* of a simplex is the convex hull of a subset of its vertices, and a *facet* of a simplex is the convex hull of all but one of its vertices. So every k-simplex has k + 1 facets and 2^k faces.
- A *geometric simplicial complex* is a set Delta of simplices in some R^d satisfying two conditions:
  1. every face of a simplex in Delta is also in Delta
  2. the intersection of two simplices in Delta is a face of both. i.e., the simplicies in Delta have disjoint interiors
- The simplicies in Delta are called its *cells*.
- The *underlying space* of Delta is the union of its simplices. You can think of Delta as a triangulation of its underlying space.
- A *geometric realization* of an abstract simplicial complex calX adds maps vertices of calX to points so that its simplices form a geometric simplicial complex.
- Menger showed any d-dimensional abstract simplicial complex can be geometrically realized in R^{2d + 1}. For example, any simple graph can be embedded in R^3.
- We can naturally generalize these ideas to use arbitrary polytopes instead of just simplicies to create *polytopal complexes*. For example, we could glue a bunch of k-dimensional cubes together instead of simplices.

## Delta-Complexes

- These geometric simplicial complexes nicely match intuition for how a triangulation of a geometric space should behave, but they're a bit more restrictive than we might like.
- You have to have an embedding in R^d, so a triangulation of an abstract 2-manifold is not a geometric simplicial complex. Also, each k-simplex has k + 1 distinct vertices and each subset of k + 1 vertices leads to at most one k-simplex. Both factors rule out constructions like a system of loops.
- We'll use *Delta-complexes* to represent something a bit more general: intuitively, we want to create topological spaces by gluing simplices together along their boundary, without needing an actual embedding in an ambient space.
- The things we're going to glue together are the following: the *standard n-simplex* Delta_n is the convex hull of the positive unit coordinate vectors in R^{n + 1}:

    - Delta_n := {(lambda_0, lambda_1, …, lambda_n) | \sum_{i = 0}^n lambda_i = 1 and 0 ≤ lambda_i ≤ 1 for all i}
- A more general *n-simplex* is the image of Delta_n under a homeomorphism. Every k-dimensional face of Delta_n is a k-simplex.
- For any integer -1 ≤ k ≤ n, we let Delta_n ^(k) denote the union of all (n choose k + 1) k-dimensional faces of Delta_n. In particular, Delta_n^(n - 1) is the *boundary* of Delta_n, also denoted as partial Delta_n.
- To define a Delta-complex X, we consider an inductively defined sequence of topological spaces X^(0) subseteq X^(1) subseteq … X^(n) = X. Each space X^(k) is called the *k-skeleton* of X.
- The 0-skeleton is a discrete set of points called the *vertices* (again).
- For each k > 0, we construct the k-skeleton X^(k) by attaching k-simplices to the (k-1) skeleton. Each k-simplex Delta_k is attached by a *gluing map* sigma : partial Delta_k ➡ X(k - 1) that maps the interior of each face of Delta_k homeomorphically to the interior of a simplex in X^(k - 1) of the same dimension.
- Inductively, we see sigma mapping each vertex of Delta_k to a point in X^(0), each edge to an edge of X^(1), and so on. The interiors of the k-simplicies are also called the *k-cells* of X.
- Delta-complex X is *regular* if every k-cell in X has k + 1 distinct vertices for all k (i.e., no loops). A regular Delta-complex is *proper* (or *simplicial*) if each subset of k + 1 vertices is incident to at most one k-cell for all k.
- Equivalently, the intersection of any two cells in a proper Delta-complex is a face of both cells. We can then prove that every proper Delta-complex is homeomorphic to a geometric simplicial complex via a homeomorphism that maps simplicies to simplicies. In turn, the term *simplicial complex* may refer either to a geometric simplicial complex or to a proper Delta-complex.

## CW-complexes (maybe skip)

- Of course, we still can't create a system of loops with what we've seen so far.
- One thing we can do is replace simplices in Delta-complexes with arbitrary polytopes to obtain *polytope complexes* that use cellular gluing maps which map the interior of each k-dimensional face of a cell homomorphically to the interior of a k-dimensional cell.
- But while we're here, we may as well describe the most general notion of a cell complex by ignoring the facial structure. Here, we'll have what Whitehead originally called membrane complexes, but are now called *closure finite complexes with weak topology* or *CW complexes* for short.
- We no longer need a facial structure, so instead of simplicies or even polytopes, we're going to work with standard unit ball B^n := {x in R^n | ||X|| ≤ 1}. The boundary of the unit ball B^n is the standard unit sphere S^{n - 1} := {x in R^n | ||X|| ≤ 1}.

- Again, we use a nest sequence of topological spaces X^(0) subseteq X^(1) subseteq … X(n). Again, the 0-skeleton is a discrete set of points.
- For each k > 0, the k-skeleton X^(k) is the result of attaching a set of k-dimensional balls to X(k - 1) by gluing maps of the form sigma : S^{k - 1} → X^(k - 1). Unlike with Delta-complexes, I'm not requiring the gluing map to "fill" entire (n-1)-dimensional balls. The interior of the balls B^k attached to X^(k - 1) are the k-cells of X.
- A CW complex is *regular* if its gluing maps are embeddings (homeomorphisms onto their images) and further its *proper* if the intersection of any two closed cells is another closed cell.

## A Few More Examples

- So that's a lot of generalizations of similar ideas. But we've seen many of these ideas before!
- Simple graphs are 1-dimensional (abstract) simplicial complexes.
- Graphs with parallel edges but no loops are regular Delta-complexes. Yes loops and you have non-regular Delta-complexes.
- Polygonal schemata are 2-dimensional polytope complexes and also 2-dimensional CW complexes.
- Next lecture, we'll go over a few more examples of complexes that are useful for understanding sets of data points.