Main topics are #cell_complex.

Cell Complexes

- We’ve talked a lot about 2-manifolds and representing them with graph embeddings. Today we’re finally going to generalize this idea to higher dimensions by talking about cell complexes.
- A cell complex is a decomposition of a topological space into pieces of simple topology that are glued together ‘nicely’ along their boundaries. There are many kinds that all have different definitions. So let’s go!
- Let’s start general to establish some common terminology before we swing to the other extreme.
- An abstract simplicial complex is a collection $\mathcal{X}$ of finite sets closed under taking subsets.
- The sets in $\mathcal{X}$ are called simplices. The vertices of $\mathcal{X}$ are the singleton sets (or their elements) denoted as $\mathcal{X}_0$.
- The dimension of a simplex is one less than its cardinality (so vertices have dimension 0) and the dimension of the complex is the largest dimension of any simplex.
- Subsets of a simplex are its faces. A simplex is maximal or a facet if it is not a proper face of any other simplex. Finally, $\mathcal{X}$ is pure if every facet has the same dimension.
- Some examples:
  - Power set of any finite set
  - Simple undirected graphs with no isolated vertices are pure 2-dim ASCs.

Geometric Simplicial Complexes

- But we need something a bit more concrete to hang out hat onto.
- It turns out any abstract simplicial complex $\mathcal{X}$ has a geometric realization, a map of $\mathcal{X}$ to generic points in a sufficiently high dimensional space.
- The convex hull of a finite set $X$ subset $\mathbb{R}^d$ is the set of all weighted averages of points in $X$.
  - $\text{conv}(X) := \{\sum_{x \in X} \lambda_x x | \sum_{x \in X} \lambda_x = 1 \text{ and } 0 \leq \lambda_x \leq 1 \text{ for all } x \in X\}$
- The convex hull of a finite set of points is called a polytope. A proper face of a polytope is its intersection with any hyperplane that does not intersect the interior. Any proper face of a polytope is also a polytope.
- Say $k$ points in $\mathbb{R}^d$ are affinely independent if their convex hull is $k$-dimensional.
• A *k-simplex* is the convex hull of a set of $k + 1$ affinity independent points called its vertices.

  ![A 3-simplex, a 2-simplex, a 1-simplex, a 0-simplex, and the (-1)-simplex](image)

• A *face* of a simplex is the convex hull of a subset of its vertices, and a *facet* of a simplex is the convex hull of all but one of its vertices. So every $k$-simplex has $k + 1$ facets and $2^k$ faces.

• A *geometric simplicial complex* is a set $\Delta$ of simplices in some $\mathbb{R}^d$ satisfying two conditions:
  1. every face of a simplex in $\Delta$ is also in $\Delta$
  2. the intersection of two simplices in $\Delta$ is a face of both. i.e., the simplicies in $\Delta$ have disjoint interiors

  - The simplicies in $\Delta$ are called its *cells*.
  - The *underlying space* of $\Delta$ is the union of its simplices. You can think of $\Delta$ as a triangulation of its underlying space.
  - A *geometric realization* of an abstract simplicial complex $\mathcal{X}$ adds maps vertices of $\mathcal{X}$ to points so that its simplices form a geometric simplicial complex.
  - Menger showed any $d$-dimensional abstract simplicial complex can be geometrically realized in $\mathbb{R}^{2d + 1}$. For example, any simple graph can be embedded in $\mathbb{R}^3$.
  - We can naturally generalize these ideas to use arbitrary polytopes instead of just simplicies to create *polytopal complexes*. For example, we could glue a bunch of $k$-dimensional cubes together instead of simplices.

**Delta-Complexes**

• These geometric simplicial complexes nicely match intuition for how a triangulation of a geometric space should behave, but they’re a bit more restrictive than we might like.

• You have to have an embedding in $\mathbb{R}^d$, so a triangulation of an abstract 2-manifold is not a geometric simplicial complex. Also, each $k$-simplex has $k + 1$ distinct vertices and each subset of $k + 1$ vertices leads to at most one $k$-simplex. Both factors rule out constructions like a system of loops.

• We’ll use *Delta-complexes* to represent something a bit more general: intuitively, we want to create topological spaces by gluing simplices together along their boundary, without needing an actual embedding in an ambient space.

• The things we’re going to glue together are the following: the *standard $n$-simplex* $\Delta_n$ is the convex hull of the positive unit coordinate vectors in $\mathbb{R}^{n+1}$:
• Delta_n := \{(\lambda_0, \lambda_1, \ldots, \lambda_n) \mid \sum_{i = 0}^n \lambda_i = 1 \text{ and } 0 \leq \lambda_i \leq 1 \text{ for all } i\}

• A more general n-simplex is the image of Delta_n under a homeomorphism. Every k-dimensional face of Delta_n is a k-simplex.

• For any integer -1 \leq k \leq n, we let Delta_n \wedge(k) denote the union of all \((n \text{ choose } k + 1)\) k-dimensional faces of Delta_n. In particular, Delta_n \wedge(n - 1) is the boundary of Delta_n, also denoted as partial Delta_n.

• To define a Delta-complex X, we consider an inductively defined sequence of topological spaces X^(0) \subseteq X^(1) \subseteq \ldots X^(n) = X. Each space X^(k) is called the k-skeleton of X.

• The 0-skeleton is a discrete set of points called the vertices (again).

• For each k > 0, we construct the k-skeleton X^(k) by attaching k-simplices to the (k-1) skeleton. Each k-simplex Delta_k is attached by a gluing map \(\sigma : \text{partial Delta}_k \rightarrow X(k - 1)\) that maps the interior of each face of Delta_k homeomorphically to the interior of a simplex in X^(k - 1) of the same dimension.

• Inductively, we see \(\sigma\) mapping each vertex of Delta_k to a point in X^(0), each edge to an edge of X^(1), and so on. The interiors of the k-simplices are also called the k-cells of X.

• Delta-complex X is regular if every k-cell in X has k + 1 distinct vertices for all k (i.e., no loops). A regular Delta-complex is proper (or simplicial) if each subset of k + 1 vertices is incident to at most one k-cell for all k.

• Equivalently, the intersection of any two cells in a proper Delta-complex is a face of both cells. We can then prove that every proper Delta-complex is homeomorphic to a geometric simplicial complex via a homeomorphism that maps simplicies to simplicies. In turn, the term simplicial complex may refer either to a geometric simplicial complex or to a proper Delta-complex.

**CW-complexes (maybe skip)**

• Of course, we still can’t create a system of loops with what we’ve seen so far.

• One thing we can do is replace simplices in Delta-complexes with arbitrary polytopes to obtain polytope complexes that use cellular gluing maps which map the interior of each k-dimensional face of a cell homomorphically to the interior of a k-dimensional cell.

• But while we’re here, we may as well describe the most general notion of a cell complex by ignoring the facial structure. Here, we’ll have what Whitehead originally called membrane complexes, but are now called closure finite complexes with weak topology or CW complexes for short.

• We no longer need a facial structure, so instead of simplicies or even polytopes, we’re going to work with standard unit ball B^n := \{x in R^n \mid \|X\| \leq 1\}. The boundary of the unit ball B^n is the standard unit sphere S^(n - 1) := \{x in R^n \mid \|X\| \leq 1\}.
• Again, we use a nest sequence of topological spaces $X^0 \subseteq X^1 \subseteq \ldots \subseteq X^n$. Again, the 0-skeleton is a discrete set of points.

• For each $k > 0$, the $k$-skeleton $X^k$ is the result of attaching a set of $k$-dimensional balls to $X^{k-1}$ by gluing maps of the form $\sigma : S^{k-1} \to X^{k-1}$. Unlike with Delta-complexes, I’m not requiring the gluing map to “fill” entire $(n-1)$-dimensional balls. The interior of the balls $B^k$ attached to $X^{k-1}$ are the $k$-cells of $X$.

• A CW complex is **regular** if its gluing maps are embeddings (homeomorphisms onto their images) and further its **proper** if the intersection of any two closed cells is another closed cell.

**A Few More Examples**

• So that’s a lot of generalizations of similar ideas. But we’ve seen many of these ideas before!

• Simple graphs are 1-dimensional (abstract) simplicial complexes.

• Graphs with parallel edges but no loops are regular Delta-complexes. Yes loops and you have non-regular Delta-complexes.

• Polygonal schemata are 2-dimensional polytope complexes and also 2-dimensional CW complexes.

• Next lecture, we’ll go over a few more examples of complexes that are useful for understanding sets of data points.