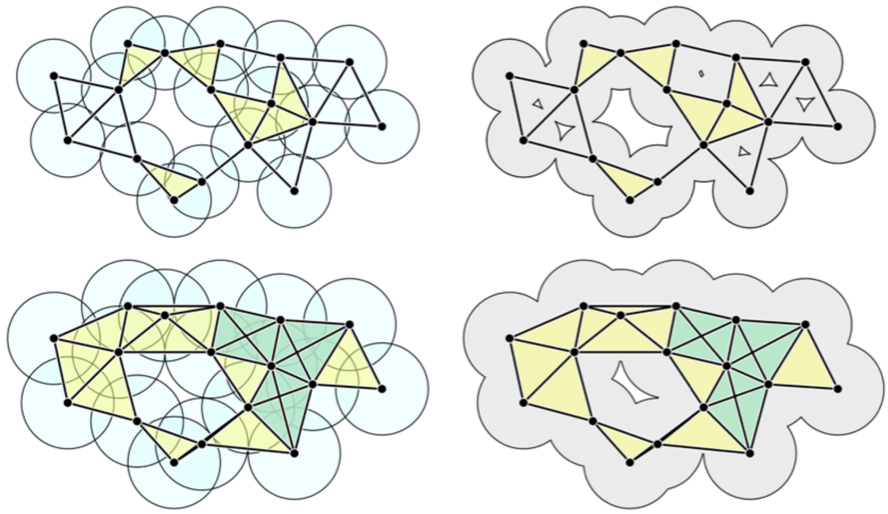
# CS 7301.003.20F Lecture 21—October 28, 2020

Main topics are #cell_complex_examples.

- Today, we're going to consider the following scenario: You're given a point cloud representing some kind of data. By itself, the cloud does not provide much information, but if we can impose some kind of natural topology onto the cloud, then maybe we can learn something.
- Most strategies of this sort try to connect points that are sufficiently 'close' together. The intuition being that if the point cloud represents a 'dense' sampling of sufficiently 'nice' domain, we might discover the domain through these connections.

## Aleksandrov-Čech Complexes

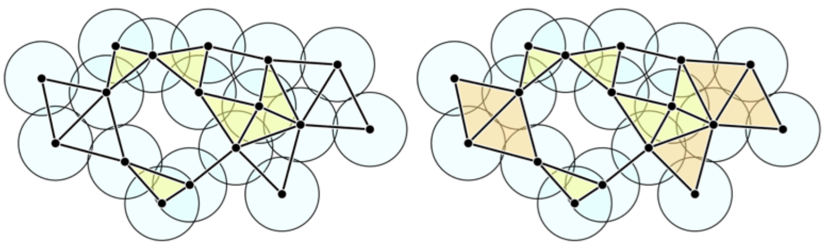
- Let P be a set of points in some metric space S such as R^d, and let eps be a positive real number.
- The Aleksandrov-Čech complex AČ_eps(P) (sometimes just called the Čech complex) is the *intersection complex* or *nerve* of a set of balls of radius eps centered at points of P.
- This means k + 1 points of P define a k-simplex in AČ_eps(P) if and only if the eps-balls centered at those points have a non-empty common intersection.
- Equivalently, the k + 1 points lie in a single ball of radius eps.
- It's formally an abstract simplicial complex of arbitrarily high dimension, because the simplices overlap.
- The AČ complex captures almost all of the topology of the union of eps-balls.
- To make this precise, we say two topological spaces X and Y are *homotopy equivalent* if there exists a pair of continuous functions f : X → Y and g : Y → X such that g circ f and f circ g are homotopic to the identify functions in their respective domains. Intuitively, it means both spaces can be continuously deformed to look like the other.
- Leray ['45] proved the following very important result.
- The Nerve Lemma: Let calU = {U_1, …, U_n} be a finite set of open sets, such that the intersection of any subset of calU is either empty or contractible. Then the nerve of calU is homotopy equivalent to the union of sets in calU.
- Corollary: For any point set P and radius eps, the Aleksandrov-Čech complex AČ_eps(P) is homotopy equivalent to the union of balls of radius eps centered at points in P.



Aleksandrov-Čech complexes and unions of balls for two different radii. 2-simplices are yellow; 3-simplices are green.

## Vietoris-Rips Complexes

- Let's look at a slightly "simpler" complex. First, the *proximity graph* N_eps(P) of P is the geometric graph with vertices P and edges joining pairs of vertices at distance at most 2eps; in other words N_eps(P) is the 1-skeleton of AČ_eps(P).
- The Vietoris-Rips complex VR_eps(P) is the *flag complex* or *clique complex* of N_eps(P).
- This means k + 1 points of P define a k-simplex in VR_eps(P) if and only if every pair of those points defines an edge in N_eps(P).
- Equivalently, the k + 1points have diameter at most 2eps.
- Again, it is an abstract simplicial complex.
- The triangle inequality implies AČ_eps(P) subseteq VR_eps(P) subseteq AČ_{2eps}(P).

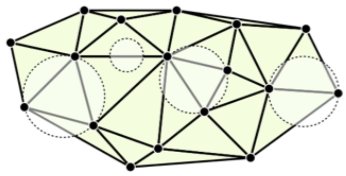


The Aleksandrov-Čech and Vietoris-Rips complexes for the same points and radius.

- One advantage we have working with Vietoris-Rips complexes is that we only need the proximity graph to determine them. That's useful in certain contexts like sensor networks where we may not know the underlying metric.
- Unfortunately, though, there is no result like the Nerve Lemma for flag complexes, and you can even find sets of points in the plane where the Vietoris-Rips complex has topological features of arbitrarily high dimension.
- There are a couple weaker results found within the last 20 years, though. See Erickson's notes for some examples.

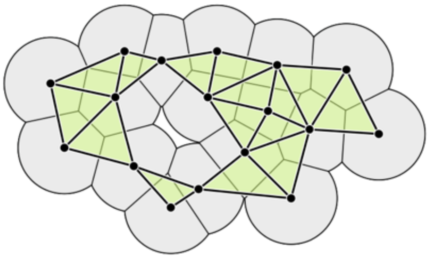
## Delaunay and Alpha Complexes

- Both examples so far have been abstract simplicial complexes. Let's go for something geometric this time. If you've taken a computational geometry class, what I'm about to say may sound familiar.
- A *Delaunay ball* for P is a closed ball that has no points of P in its interior.
- For any Delaunay ball B, the convex hull of B cap P = partial B cap P is called a *Delaunay cell* for P.
- They're convex hulls, so each cell is a convex polytope. Also, the intersection of two Delaunay cells is a face of both.
- Therefore, the Delaunay cells define a polytopal complex called the Delaunay complex.
- If the points are in general position, meaning no more than d + 2 lie on a sphere, then every maximal Delaunay cell is a d-simplex. The Delaunay complex is therefore a geometric simplicial complex called the *Delaunay triangulation*.

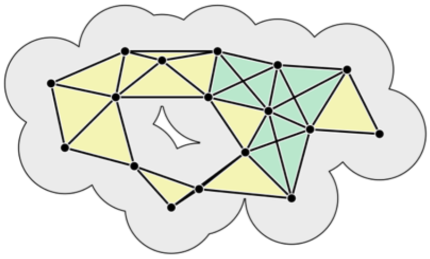


A Delaunay triangulation, with four Delaunay balls emphasized.

- The union of cells in a the Delaunay complex of P is the convex hull of P, but now we're left with just a very simple topology.
- So fix a real radius eps > 0 as before. For each point p in P, let B_eps(p) denote the set of points in the underlying space whose nearest neighbor in P is p and whose distance to p is at most eps / 2.
- In other words, B_eps(p) is the intersection of the eps-ball centered at p and the *Voronoi region* of p.
- The *alpha complex* alpha_eps(P) is the intersection complex of the set {B_eps(p) | p in P}.
- Again, this is a polytopal or geometric simplicial complex.
- The underlying space |alpha_eps(P)| of the alpha complex is called an *alpha shape* of P.
- The Nerve Lemma immediately implies the alpha shape is homotopy equivalent to the union of the eps-balls, but now we're avoiding those higher dimensional simplicies.
- If point set P is in general position, then the alpha complex alpha_eps(P) is the intersection of the Delaunay triangulation of P and the Aleksandrov-Čech complex AČ_eps(P) (this may not be surprising if you recall the duality between Voronoi diagrams and Delaunay triangulations).
- In other words, k + 1 points in P define a simplex in the alpha complex if and only if they lie in a closed ball B with diameter at most eps that contains no other point in P.



An alpha complex and a decomposed union of balls.



The corresponding Aleksandrov-Čech complex.